

Exterior Algebra Structure for Relative Invariants of Reflections Groups

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Abstract

Let G be a reflection group acting on a vector space V (over a field with zero characteristic). We denote by $S(V^*)$ the coordinate ring of V , by M a finite dimensional G -module and by χ a one-dimensional character of G . In this article, we define an algebra structure on the isotypic component associated to χ of the algebra $S(V^*) \otimes \Lambda(M^*)$. This structure is then used to obtain various generalizations of usual criterions on regularity of integers.

Keywords. Reflection Group - Relative Invariant - Exterior Algebra - Regular Integer - Hyperplane Arrangement

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1 Introduction

In the first part of this article, we will study the following situation. Let G be a reflection group acting on the vector space V , M be a finite dimensional representation of G and χ be a one-dimensional character of G . Following the ideas of Shepler [14], we construct an exterior algebra structure on the χ -isotypic component of $T^{-1}S(V^*) \otimes \Lambda(M^*)$ for a suitable multiplicative set T of $S(V^*)$. This work is in line with the articles [11], [14], [7] and [1] which construct algebra structures on the χ -isotypic of the algebra $S(V^*) \otimes \Lambda(M^*)$ under conditions over the restrictions of M and χ to certain subgroups of G . Here, the idea is to transfer the hypotheses on M and χ to conditions on the base ring : we substitute $S(V^*)$ in a bigger ring (a fraction ring of $S(V^*)$) in which some linear forms associated to hyperplanes of G are invertible. The conditions will be held by the "bad" hyperplanes that are needed to be invertible. The main results of [11], [14], [7] et [1] are exceptional cases of proposition 25 (see remark 26). The article [6] explains the situation in prime characteristic.

In the second part of this article, we will give consequences of the exterior algebra structure with links to the notion of regular integers. These consequences are similar to those that can be found in [11], [8], [7] and [3].

Various types of hyperplanes appear in the first part of the article. The hyperplanes to invert (the multiplicative set T) are chosen following these types. The third part studies these types for concrete reflection groups : the symmetric group, $G(de, e, 2)$, $G(d, 1, r)$ and the exceptional group G_4 , G_5 and G_{24} .

Let us begin with some usual definitions and notations.

Definition 1 – Reflection. Let k be a field of characteristic 0 and V a finite dimensional vector space over k . Any $g \in \text{GL}(V)$ so that g is of finite order and $\ker(g - 1)$ is an hyperplane of V is called a *reflection*.

Definition 2 – Reflection Groups. Let k be a field of characteristic 0. (G, V) is said to be a *reflection group over k* if V is a finite dimensional vector space over k (we denote by ℓ the dimension of V) and G be a finite subgroup of $\text{GL}(V)$ generated by reflections. It will often be more comfortable to write "let G be a reflection group" omitting the vector space V .

Notation 3 – Reflection and hyperplane. Let (G, V) be a reflection group. We denote by \mathcal{S} the set of reflections of G and \mathcal{H} the set of hyperplanes of G :

$$\mathcal{S} = \{s \in G, \dim \ker(s - \text{id}) = \dim V - 1\} \quad \text{and} \quad \mathcal{H} = \{\ker(s - \text{id}), \quad s \in \mathcal{S}\}.$$

Notation 4 – Around a hyperplane. Let (G, V) be a reflection group. For $H \in \mathcal{H}$,

- one chooses $\alpha_H \in V^*$ a linear form with kernel H ;
- one sets $G_H = \text{Fix}_G(H) = \{g \in G, \forall x \in H, gx = x\}$. This is a cyclic subgroup of G . We denote by e_H its order and by s_H its generator with determinant $\zeta_H = \exp(2i\pi/e_H)$;
- For any finite dimensional kG -module N and for $j \in \llbracket 0, e_H - 1 \rrbracket$, we define the integers $n_H(N)$ and $n_{j,H}(N)$ by

$$\text{Res}_{G_H}^G(N) = \bigoplus_{j=0}^{e_H-1} n_{j,H}(N) \det^{-j} \quad \text{and} \quad n_H(N) = \sum_{j=0}^{e_H-1} j n_{j,H}(N);$$

the integer $n_{j,H}(N)$ is nothing else but the multiplicity of ζ_H^j as an eigenvalue of s_H acting on N^* ;

- let $\chi : G \rightarrow k^\times$ be a linear character of G , we denote by k_χ the representation of G with character χ over k and $n_H(\chi)$ for $n_H(k_\chi)$; by definition, $n_H(\chi)$ is the unique integer j verifying $0 \leq j < e_H$ and $\chi(s_H) = \det(s_H)^{-j}$. Finally, for any kG -module N , we denote by $N^\chi = \{x \in N, gx = \chi(g)x\}$ the χ -isotypic component of N .

Definition 5 – Polynomial function associated to a representation. Let (G, V) be a reflection group and N be a finite dimensional G -module. We set

$$Q_N = \prod_{H \in \mathcal{H}} \alpha_H^{n_H(N)} \in S(V^*).$$

When χ is a linear character of G , we set Q_χ rather than Q_{k_χ} , so that

$$Q_\chi = \prod_{H \in \mathcal{H}} \alpha_H^{n_H(\chi)} \in S(V^*).$$

2 Construction of the Algebra Structure

In this section and the next one, we fix a reflection group (G, V) over k , a kG -module M with dimension r and $\chi : G \rightarrow k^\times$ a linear character of G . We denote by \det_M (resp. \det_{M^*}) the determinant of the representation M (resp. M^*).

Notation 6 Let $\mathcal{B} \subset \mathcal{H}$ be a G -stable subset (which will be the "bad" hyperplanes), we denote by $\mathcal{G} = \mathcal{H} \setminus \mathcal{B}$ and $T = \langle \alpha_H, \quad H \in \mathcal{B} \rangle_{\text{mult. set}}$ the multiplicative subset of $S(V^*)$

associated to \mathcal{B} . We then set $\Omega = T^{-1}S(V^*) \otimes \Lambda(M^*)$ and $\Omega^p = T^{-1}S(V^*) \otimes \Lambda^p(M^*)$ for $p \in \llbracket 0, r \rrbracket$. Thus we have

$$\Omega^\chi = \bigoplus_{p=0}^r (\Omega^p)^\chi.$$

As said in the introduction, the idea of the roof is to bring the "bad" hyperplanes together in the subset \mathcal{B} . Thus in subsection 2.1, we begin to define some "hyperplane types" so that we are able to differentiate the hyperplanes of \mathcal{H} . In subsection 2.2, we construct the algebra structure on Ω^χ . Finally, in subsection 2.3, we study this algebra structure.

2.1 Hyperplanes

To define the notion of (M, χ) -acceptable hyperplane, we need to introduce a notation which will also be useful for the next subsection.

Notation 7 For $H \in \mathcal{H}$, we denote by (j_1, \dots, j_r) a family of integers such that for all $i \in \llbracket 1, r \rrbracket$ we have $0 \leq j_i \leq e_H - 1$ and $\text{Res}_{G_H}^G(M) = \det^{-j_1} \oplus \dots \oplus \det^{-j_r}$. The family (j_1, \dots, j_r) is not unique but so is the multi-set associated to it. The integers j_i are closely related to the integers $n_{j,H}(M)$ (see notation 4). Precisely, for $j \in \llbracket 0, e_H - 1 \rrbracket$, $n_{j,H}(M)$ is the number of $i \in \llbracket 1, r \rrbracket$ so that $j_i = j$, so that we have

$$\sum_{i=1}^r j_i = \sum_{j=0}^{e_H-1} j n_{j,H}(M) = n_H(M).$$

In addition, the family (j_1, \dots, j_r) is so that the eigenvalues of s_H acting on M^* are the $\zeta_H^{j_i}$ for $i \in \llbracket 1, r \rrbracket$.

We define four types of hyperplanes (two types associated with a linear representation of G and two others associated with a couple constituted of a representation of G and a one-dimensional character of G). The M -excellent hyperplanes or the (M, χ) -good hyperplanes will be those that we do not need to inverse (see hypotheses 21 and 22).

Definition 8 – Hyperplane types. Let $H \in \mathcal{H}$; H is said to be

- (i) M -good if $n_H(M) < e_H$ and M -bad else;
- (ii) M -excellent if s_H acts on M as a reflection;
- (iii) (M, χ) -good if $n_H(M) + n_H(\chi) < e_H$;
- (iv) (M, χ) -acceptable if for all partition of the set $\llbracket 1, r \rrbracket$ in two disjoint sets (denoted respectively by I_1 and I_2), we have

$$e_H - n_H(\chi) > \sum_{i \in I_1} j_i \quad \text{or} \quad e_H - n_H(\chi) > \sum_{i \in I_2} j_i.$$

In the next remark, we study the links between the preceding notions of hyperplane types.

Remark 9 – Good and excellent hyperplanes. Let $H \in \mathcal{H}$. We denote by 1 the trivial character on G . Let show the following properties.

- (i) H is M -excellent if and only if H is M -good and M^* -good.
- (ii) If H is (M, χ) -good then H is M -good.
- (iii) H is M -excellent if and only if H is (M, χ) -acceptable for all $\chi \in \text{Hom}_{\text{gr}}(G, k^\times)$.
- (iv) If H is (M, χ) -good then H is (M, χ) -acceptable.
- (v) H is M -good if and only if H is $(M, 1)$ -good.
- (vi) If H is M -good then H is $(M, 1)$ -acceptable.

Let us begin with (i). We start to express $n_H(M^*)$ using $n_H(M)$:

$$n_H(M^*) = \sum_{j=1}^{e_H-1} (e_H - j)n_{j,H}(M) = e_H(r - n_{0,H}(M)) - n_H(M).$$

We have $n_H(M) = 0$ if and only if $n_H(M^*) = 0$ if and only if s_H acts trivially on M . We then deduce

$$n_H(M) \leq e_H - 1 \quad \text{and} \quad n_H(M^*) \leq e_H - 1 \quad \Longleftrightarrow \quad r - n_{0,H}(M) < 2.$$

Since $n_{0,H}(M)$ (resp. $n_{0,H}(M^*)$) is the multiplicity of 1 as an eigenvalue of s_H acting on M (resp. M^*), we obtain $n_{0,H}(M) = n_{0,H}(M^*)$ and the condition $n_{0,H}(M) \in \{r-1, r\}$ can be expressed geometrically as s_H acts trivially on M or acts as a reflection on M . In particular, an M -excellent hyperplane is always M -good.

Let us show (ii). We have $n_H(\chi) \geq 0$, so an (M, χ) -good hyperplane is always M -good.

Now, let us consider (iii). Let us assume that H is an M -excellent hyperplane. There exists at most one $i_0 \in \llbracket 1, r \rrbracket$ so that j_{i_0} is nonzero. When I_1 and I_2 are two disjoint subsets of $\llbracket 1, r \rrbracket$, only one of those two sets can contain i_0 . Thus, we have

$$\sum_{i \in I_1} j_i = 0 < e_H - n_H(\chi) \quad \text{or} \quad \sum_{i \in I_2} j_i = 0 < e_H - n_H(\chi).$$

Reciprocally, let us assume that H is not M -excellent. We then deduce that there exists $i_1 \neq i_2$ so that $j_{i_1} \neq 0$ and $j_{i_2} \neq 0$. In addition, we know the existence of a linear character χ of G so that $n_H(\chi) = e_H - 1$ by Stanley's theorem [16]. The disjoint sets $I_1 = \{i_1\}$ and $I_2 = \{i_2\}$ verify

$$\sum_{i \in I_1} j_i \geq 1 = e_H - n_H(\chi) \quad \text{and} \quad \sum_{i \in I_2} j_i \geq 1 = e_H - n_H(\chi).$$

We then deduce that H is not (M, χ) -acceptable.

Let us show (iv). Let I be a subset of $\llbracket 1, r \rrbracket$ so that

$$e_H - n_H(\chi) \leq \sum_{i \in I} j_i.$$

Any such I contains every $i \in \llbracket 1, r \rrbracket$ so that $j_i \neq 0$ since

$$\sum_{i \in \llbracket 1, r \rrbracket} j_i = n_H(M) \leq e_H - n_H(\chi).$$

Thus, any set I' disjoint of I verifies

$$\sum_{i \in I'} j_i = 0 < e_H - n_H(\chi).$$

Let us show (v) and (vi). Since $n_H(1) = 0$ for all $H \in \mathcal{H}$, an hyperplane is M -good if and only if it is $(M, 1)$ -good. (iv) shows that such an hyperplane is $(M, 1)$ -acceptable. \blacksquare

Since lots of hyperplanes of reflections groups verify $e_H = 2$ (for example this is the case for Coxeter groups but not only), we focus on this specific case.

Remark 10 – Hyperplanes with $e_H = 2$. Let $H \in \mathcal{H}$ so that $e_H = 2$.

Then H is M -good if and only if H is M -excellent (that is if the multiplicity of the eigenvalue -1 of s_H acting on M is not bigger than one).

If $\chi(G_H) \neq 1$ then

- (i) H is (M, χ) -acceptable if and only if s_H acts on M as a reflection or acts trivially on M (that is if the multiplicity of the eigenvalue -1 of s_H acting on M is not bigger than 1);
- (ii) H is (M, χ) -good if and only if s_H acts trivially on M (that is if the multiplicity of the eigenvalue -1 of s_H acting on M is zero);

If $\chi(G_H) = 1$ then

- (i) H is (M, χ) -acceptable if and only if the multiplicity of the eigenvalue -1 of s_H acting on M is not bigger than 3;
- (ii) H is (M, χ) -good if and only if s_H acts on M as a reflection or acts trivially on M (that is if the multiplicity of the eigenvalue -1 of s_H acting on M is not bigger than 1).

Following the remark 9, it is enough to show that if H is M -good then H is M -excellent. By hypothesis, we have $n_H(M) = n_{1,H}(M) < 2$ and thus $n_{0,H}(M) = r - n_{1,H}(M) \in \{r, r-1\}$.

Let us assume that $\chi(G_H) \neq 1$. We have $n_H(\chi) \neq 0$ and then $n_H(\chi) = 1$. The definition of "acceptability" shows us that if H is (M, χ) -acceptable then H is (M, χ') -acceptable for all linear characters χ' verifying $n_H(\chi') \leq 1$. But every linear character χ' verifies $n_H(\chi') \leq 1$, thus we have H is (M, χ') -acceptable for all linear characters χ' of G . The remark 9 shows that H is M -excellent. In addition, by definition of (M, χ) -good, H is (M, χ) -good if and only if $n_H(M) < 1$ that is $n_H(M) = 0$.

Let us assume that $\chi(G_H) = 1$. We have $n_H(\chi) = 0$ and then H is (M, χ) -good if and only if $n_H(M) < 2$ (that is H is M -good). In addition, in our case, we have $j_i \in \{0, 1\}$, the multiplicity of -1 as an eigenvalue of s_H is the number of i so that $j_i \neq 0$. If they are more than 4, we can divide them in two sets of two elements and the hyperplane is not (M, χ) -good. If they are not more than 4, two disjoint set of $\llbracket 1, r \rrbracket$ cannot both contain two integers i so that $j_i \neq 0$ and finally H is (M, χ) -acceptable. ■

2.2 Construction of an Algebra Structure

Strictly following the ideas of Shepler [14], we construct an algebra structure on Ω^χ . The first step is to define a product. For this, we use the polynomial Q_χ of definition 5 to bring back the usual product of two elements of Ω^χ into Ω^χ (by Stanley theorem [16], Q_χ is so that $S(V^*) = Q_\chi S(V^*)^G$). Thus we are looking for divisibility conditions by Q_χ or more precisely by the non invertible part of Q_χ : this is done in lemmas 12 and 13. The wanted divisibility is obtained under hypotheses on \mathcal{B} (hypotheses 11 and 14).

Hypothesis 11 Let us assume that \mathcal{B} contains every M -bad hyperplane. Equivalently, every hyperplane contained in \mathcal{G} is M -good.

2.2.1 Divisibility

To begin with, let us extend the following result of divisibility [5, lemma 1] to the ring $T^{-1}S(V^*)$.

Lemma 12 — Divisibility in $T^{-1}S(V^*)$. Let $x \in T^{-1}S(V^*)$, $H \in \mathcal{H}$ and $i \in \llbracket 1, e_H \rrbracket$. Let us assume that $s_H x = \zeta_H^i x$. Then x is divisible by $\alpha_H^{e_H-i}$.

The lemma is interesting only for $H \in \mathcal{G}$ since for $H \in \mathcal{B}$, the linear form α_H is invertible in $T^{-1}S(V^*)$.

Proof. Since T is G -stable, we can write $x = P/Q$ with $Q \in T^G$. Since $S(V^*)$ is an integral domain, we deduce that $s_H P = \zeta_H^i P$. The lemma 1 of [5] shows that P is divisible by $\alpha_H^{e_H-i}$ and so is x . ■

We continue our study of divisibility by the α_H . Let us consider the case of Ω .

Lemma 13 – Divisibility in Ω . Let us choose $\mu \in (\Omega^p)^\chi$ and $H \in \mathcal{H}$. We fix (y_1, \dots, y_r) a basis of M^* so that $s_H(y_i) = \zeta_H^{j_i} y_i$ for all $i \in \llbracket 1, r \rrbracket$ (see notation 7). We write

$$\mu = \sum_{1 \leq i_1 < \dots < i_p \leq r} \mu_{i_1, \dots, i_p} y_{i_1} \wedge \dots \wedge y_{i_p} \quad \text{avec} \quad \mu_{i_1, \dots, i_p} \in T^{-1}S(V^*).$$

For $H \in \mathcal{G}$, we have

$$\begin{aligned} \text{or } 0 \leq j_{i_1} + \dots + j_{i_p} \leq e_H - 1 - n_H(\chi) \quad & \text{and} \quad \alpha_H^{j_{i_1} + \dots + j_{i_p} + n_H(\chi)} \mid \mu_{i_1, \dots, i_p} \\ \text{or } e_H - n_H(\chi) \leq j_{i_1} + \dots + j_{i_p} \leq 2e_H - 2 - n_H(\chi) \quad & \text{and} \quad \alpha_H^{j_{i_1} + \dots + j_{i_p} + n_H(\chi) - e_H} \mid \mu_{i_1, \dots, i_p}. \end{aligned}$$

Proof. Since the family $(y_{i_1} \wedge \dots \wedge y_{i_p})_{1 \leq i_1 < \dots < i_p \leq r}$ is a $T^{-1}S(V^*)$ -basis of Ω^p , we have

$$\mu_{i_1, \dots, i_p} y_{i_1} \wedge \dots \wedge y_{i_p} \in (\Omega^p)^\chi.$$

Thus $s_H(\mu_{i_1, \dots, i_p} y_{i_1} \wedge \dots \wedge y_{i_p}) = \zeta_H^{-n_H(\chi)} \mu_{i_1, \dots, i_p} y_{i_1} \wedge \dots \wedge y_{i_p}$. In addition

$$\begin{aligned} s_H(\mu_{i_1, \dots, i_p} y_{i_1} \wedge \dots \wedge y_{i_p}) &= s_H(\mu_{i_1, \dots, i_p}) s_H(y_{i_1}) \wedge \dots \wedge s_H(y_{i_p}) \\ &= \zeta_H^{j_{i_1} + \dots + j_{i_p}} s_H(\mu_{i_1, \dots, i_p}) y_{i_1} \wedge \dots \wedge y_{i_p}. \end{aligned}$$

We then deduce $s_H(\mu_{i_1, \dots, i_p}) = \zeta_H^{-j_{i_1} - \dots - j_{i_p} - n_H(\chi)} \mu_{i_1, \dots, i_p}$. The hypothesis 11 on \mathcal{B} and \mathcal{G} tells us

$$\forall H \in \mathcal{G}, \quad 0 \leq n_H(M) = \sum_{j=1}^r j_i \leq e_H - 1.$$

Hence $2 - 2e_H \leq -j_{i_1} - \dots - j_{i_p} - n_H(\chi) \leq 0$. There are two cases to distinguish :

or $1 - e_H \leq -j_{i_1} - \dots - j_{i_p} - n_H(\chi) \leq 0$ and if we set $f_H = e_H - j_{i_1} - \dots - j_{i_p} - n_H(\chi)$, we have

$$s_H(\mu_{i_1, \dots, i_p}) = \zeta_H^{f_H} \mu_{i_1, \dots, i_p} \quad \text{with} \quad 1 \leq f_H \leq e_H;$$

the lemma 12 ensures us that μ_{i_1, \dots, i_p} is divisible by $\alpha_H^{j_{i_1} + \dots + j_{i_p} + n_H(\chi)}$;

or $2 - 2e_H \leq -j_{i_1} - \dots - j_{i_p} - n_H(\chi) \leq -e_H$ and if $f_H = 2e_H - j_{i_1} - \dots - j_{i_p} - n_H(\chi)$, we have

$$s_H(\mu_{i_1, \dots, i_p}) = \zeta_H^{f_H} \mu_{i_1, \dots, i_p} \quad \text{with} \quad 2 \leq f_H \leq e_H;$$

the lemma 12 ensures us that μ_{i_1, \dots, i_p} is divisible by $\alpha_H^{j_{i_1} + \dots + j_{i_p} + n_H(\chi) - e_H}$. ■

To assure the divisibility of Q_χ , we strengthen the hypothesis 11 by the following one.

Hypothesis 14 The subset \mathcal{B} verifies the hypothesis 11 and \mathcal{B} contains every hyperplane that is not (M, χ) -acceptable. Equivalently, every hyperplane in \mathcal{G} is (M, χ) -acceptable and M -good.

We then obtain the following result of divisibility by Q_χ which is a refinement of lemma 2 of [14].

Corollary 15 – Divisibility in Ω^χ . Let us assume hypothesis 14. For $\mu, \omega \in \Omega^\chi$, we have $Q_\chi \mid \mu \wedge \omega$.

Proof. We fix $H \in \mathcal{G}$ and we consider the same basis (y_1, \dots, y_r) of M^* of lemma 13. When $I = \{i_1, \dots, i_p\}$ is a subset of $\llbracket 1, r \rrbracket$ with $1 \leq i_1 < \dots < i_p \leq r$, we set $y_I = y_{i_1} \wedge \dots \wedge y_{i_p}$. Now, we can write

$$\mu = \sum_{I \subset \llbracket 1, r \rrbracket} \mu_I y_I \quad \text{and} \quad \omega = \sum_{I \subset \llbracket 1, r \rrbracket} \omega_I y_I \quad \text{with} \quad \mu_I, \omega_I \in T^{-1}S(V^*).$$

Hence

$$\mu \wedge \omega = \sum_{I \cap J = \emptyset} \varepsilon_{I,J} \mu_I \omega_J y_{I \cup J} \quad \text{with} \quad \varepsilon_{I,J} \in \{\pm 1\}.$$

Now, let us use lemma 13. For this, we choose two subsets I, J of $\llbracket 1, r \rrbracket$ with $I \cap J = \emptyset$.

- If $0 \leq \sum_{i \in I} j_i < e_H - n_H(\chi)$ or $0 \leq \sum_{i \in J} j_i < e_H - n_H(\chi)$ then μ_I or ω_J is divisible $\alpha_H^{n_H(\chi)}$ and then $\mu_I \omega_J$ is divisible by $\alpha_H^{n_H(\chi)}$.
- If not $\sum_{i \in I} j_i \geq e_H - n_H(\chi)$ and $\sum_{i \in J} j_i \geq e_H - n_H(\chi)$ with $I \cap J = \emptyset$ which contradicts hypothesis 14.

Hence the product $\mu \wedge \omega$ is divisible by $\alpha_H^{n_H(\chi)}$ for all $H \in \mathcal{G}$. Since the family $(\alpha_H)_{H \in \mathcal{G}}$ is constituted with elements prime to each other, $\mu \wedge \omega$ is divisible by

$$\prod_{H \in \mathcal{G}} \alpha_H^{n_H(\chi)}.$$

In addition, for $H \in \mathcal{B}$, the element $\alpha_H^{n_H(\chi)}$ is invertible in $T^{-1}S(V^*)$, we finally obtain the divisibility of $\mu \wedge \omega$ by

$$\prod_{H \in \mathcal{B}} \alpha_H^{n_H(\chi)} \prod_{H \in \mathcal{G}} \alpha_H^{n_H(\chi)} = Q_\chi.$$

2.2.2 Algebra structure

When hypothesis 14 is assumed, the corollary 15 applies. For $\mu, \omega \in \Omega^\chi$, we can define the twisted product λ by

$$\mu \lambda \omega = Q_\chi^{-1} \mu \wedge \omega \in \Omega.$$

Actually, we have $\mu \lambda \omega \in \Omega^\chi$ and thus we define a law λ on Ω^χ which gives to Ω^χ a structure of an associative $(T^G)^{-1}S(V^*)^G$ -algebra with unit element Q_χ . Now, we have to show that (Ω^χ, λ) is an exterior $(T^G)^{-1}S(V^*)^G$ -algebra. For this, we study the structure constants of (Ω^χ, λ) and we will show that they are those of an exterior algebra. To have more simple notation, we set $R = S(V^*)^G$.

2.3 Exterior Algebra

In the previous subsection, under the hypothesis 14, we have constructed an algebra structure on Ω^χ . In this subsection, we are looking for its isomorphism class. The proof is divided in two stages : first, we give a necessary and sufficient condition for the structure constants of Ω^χ to be those of an exterior algebra (proposition 18); subsequently, we show that this condition is verified (proposition 24). To this perspective, we begin to generalize Stanley's theorem to the ring $T^{-1}S(V^*)$.

Corollary 16 – Stanley's Theorem in $T^{-1}S(V^*)$. We have

$$(T^{-1}S(V^*))^\chi = (T^G)^{-1}S(V^*)^G Q_\chi \quad \text{and} \quad (\Omega^r)^\chi = (T^G)^{-1}S(V^*)^G Q_{\chi \cdot \det_M} \text{vol}_M \quad (1)$$

where vol_M is a non-zero element of $\Lambda^r(M^*)$ once for all fixed.

Proof. This is an easy consequence of the usual Stanley's theorem and of the fact that $k\text{vol}_M = \Lambda^r(M^*)$ is a linear representation of G with character \det_{M^*} . \blacksquare

Notation 17 Let U be a $(T^G)^{-1}S(V^*)$ -module. For $u, v \in U$, we denote by $u \doteq v$ if there exists $x \in ((T^G)^{-1}R)^\chi$ so that $xu = v$. In particular, u and v generate the same $(T^G)^{-1}R$ -submodule.

Proposition 18 – Necessary and sufficient condition. Let us assume hypothesis 14. For every $\omega_1, \dots, \omega_r \in (\Omega^1)^\chi$, the following propositions are equivalent :

(i) for all $p \in \llbracket 1, r \rrbracket$, the family $(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p})_{1 \leq i_1 < \cdots < i_p \leq r}$ is a $(T^G)^{-1}R$ -basis of $(\Omega^p)^\chi$;
(ii)

$$\omega_1 \wedge \cdots \wedge \omega_r \doteq Q_{\chi \cdot \det_M} \text{vol}_M. \quad (2)$$

Proof. (i) \Rightarrow (ii). This is an easy consequence of (1).

(ii) \Rightarrow (i). We set $K = \text{Frac}(T^{-1}S(V^*))$. Let us show that $\mathcal{F} = (\omega_{i_1} \wedge \cdots \wedge \omega_{i_p})_{1 \leq i_1 < \cdots < i_p \leq r}$ is free over K . Since $\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} = Q_\chi^{1-p} \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}$, it suffices to show that the family $(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p})_{1 \leq i_1 < \cdots < i_p \leq r}$ is free over K . For that, let us consider the relation

$$\sum_{1 \leq i_1 < \cdots < i_p \leq r} r_{i_1, \dots, i_p} \omega_{i_1} \wedge \cdots \wedge \omega_{i_p} = 0 \quad \text{with} \quad r_{i_1, \dots, i_p} \in K.$$

We fix $I = \{i_1, \dots, i_p\} \subset \llbracket 1, r \rrbracket$ with $1 \leq i_1 < \cdots < i_p \leq r$ and we set $I^c = \{i_{p+1}, \dots, i_r\}$ the complementary of I in $\llbracket 1, r \rrbracket$. Multiplying the preceding relation by $\omega_{i_{p+1}} \wedge \cdots \wedge \omega_{i_r}$, we obtain

$$r_{i_1, \dots, i_p} \omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \wedge \omega_{i_{p+1}} \wedge \cdots \wedge \omega_{i_r} = 0.$$

Hence $0 = r_{i_1, \dots, i_p} Q_\chi^{r-1} \omega_1 \wedge \cdots \wedge \omega_r \doteq r_{i_1, \dots, i_p} Q_\chi^{r-1} Q_{\chi \cdot \det_M} \text{vol}_M$. We then deduce that $r_{i_1, \dots, i_p} = 0$ and thus the K -freeness of the family $(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p})_{1 \leq i_1 < \cdots < i_p \leq r}$.

Finally, the family \mathcal{F} is a basis of K -vector space $K \otimes \Lambda^p(M^*)$. Thus, if $\mu \in (\Omega^p)^\chi$, there exists $r_{i_1, \dots, i_p} \in K$ so that

$$\mu = \sum_{1 \leq i_1 < \cdots < i_p \leq r} r_{i_1, \dots, i_p} \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}.$$

Let us fix again $I = \{i_1, \dots, i_p\} \subset \llbracket 1, r \rrbracket$ with $i_1 < \cdots < i_p$ and set $I^c = \{i_{p+1}, \dots, i_r\}$ its complementary. By multiplying the defining relation of μ by $\omega_{i_{p+1}} \wedge \cdots \wedge \omega_{i_r}$, we obtain $\mu \wedge \omega_{i_{p+1}} \wedge \cdots \wedge \omega_{i_r} \in (\Omega^r)^\chi$. Thus, with (1), there exists $f \in (T^G)^{-1}R$ so that

$$\mu \wedge \omega_{i_{p+1}} \wedge \cdots \wedge \omega_{i_r} = f Q_{\chi \cdot \det_M} \text{vol}_M.$$

In addition,

$$\mu \wedge \omega_{i_{p+1}} \wedge \cdots \wedge \omega_{i_r} = \varepsilon r_{i_1, \dots, i_p} \omega_1 \wedge \cdots \wedge \omega_r \doteq \varepsilon r_{i_1, \dots, i_p} Q_{\chi \cdot \det_M} \text{vol}_M \quad \text{avec} \quad \varepsilon \in \{\pm 1\}.$$

Hence $r_{i_1, \dots, i_p} \doteq \varepsilon f \in (T^G)^{-1}R$. Therefore, the family \mathcal{F} is a $(T^G)^{-1}R$ -basis of $(\Omega^p)^\chi$. \blacksquare

Now, the aim is to show that every $(T^G)^{-1}R$ -basis of $(\Omega^1)^\chi$ verifies condition 2. For this, we construct a family $(\nu_i)_{1 \leq i \leq r}$ of G -invariants and a family $(\mu_i)_{1 \leq i \leq r}$ of \det_{M^*} -invariants verifying respectively

$$\nu_1 \wedge \cdots \wedge \nu_r \doteq Q_M \text{vol}_M \quad \text{and} \quad \mu_1 \wedge \cdots \wedge \mu_r \doteq (Q_{M^*})^{r-1} \text{vol}_M.$$

Proposition 19 – Invariants in Ω^1 . There exists a family $(\nu_1, \dots, \nu_r) \in ((\Omega^1)^G)^r$ and a family $(\mu_1, \dots, \mu_r) \in ((\Omega^1)^{\det_{M^*}})^r$ verifying

$$\nu_1 \wedge \cdots \wedge \nu_r \in k^\times Q_M \text{vol}_M \quad \text{and} \quad \mu_1 \wedge \cdots \wedge \mu_r \in k^\times (Q_{M^*})^{r-1} \text{vol}_M.$$

Proof. The proof is a rephrasing of Gutkin's theorem [5] using the notion of minimal matrix evolved by Opdam in [10, definition 2.2 and proposition 2.4 (ii)].

For $C = (c_{ij})_{i,j} \in \text{M}_r(S(V^*))$, we denote by $g \cdot C$ the matrix $(g c_{ij})_{i,j}$. Let us consider C a M -minimal matrix. By definition, $C \in \text{M}_r(S(V^*))$ verifies

$$(i) \quad g \cdot C = C g_M;$$

(ii) $\det C \neq 0$;

(iii) $\deg \det C$ is minimal among the matrix verifying (i) and (ii).

Let us choose $(y_i)_{1 \leq i \leq r}$ a basis of M^* and define for $j \in \llbracket 1, r \rrbracket$,

$$\nu_j = \sum_{i=1}^r c_{ji} \otimes y_i.$$

We have $\nu_1 \wedge \cdots \wedge \nu_r \in k^\times \det(C) \text{vol}_M$. Although $\det C \in k^\times Q_M$ (let us see [10, proposition 2.4 (iii)]). It remains to show that ν_j is G -invariant for all $j \in \llbracket 1, r \rrbracket$. But, (i) gives

$$g c_{ji} = \sum_{k=1}^r c_{jk} g_{M_{ki}} \quad \text{and} \quad g y_i = \sum_{n=1}^r g_{M^*_{ni}} y_n.$$

Hence

$$g \nu_j = \sum_{i=1}^r \left(\sum_{k=1}^r c_{jk} g_{M_{ki}} \otimes \sum_{n=1}^r g_{M^*_{ni}} y_n \right) = \sum_{k=1}^r \sum_{n=1}^r \left(\sum_{i=1}^r g_{M_{ki}} g_{M^*_{ni}} \right) c_{jk} \otimes y_n.$$

Since ${}^t g_{M^*} = g_M^{-1}$, we have $\sum_{i=1}^r g_{M_{ki}} g_{M^*_{ni}} = \delta_{kn}$ and then

$$g \nu_j = \sum_{k=1}^r c_{jk} \otimes y_k = \nu_j.$$

Finally ν_j is G -invariant.

Now, let us consider D a M^* -minimal matrix. By definition, $D \in M_r(S(V^*))$ verifies

(i) $g \cdot D = D g_{M^*}$;

(ii) $\det D \neq 0$;

(iii) $\deg \det D$ is minimal among the matrix verifying (i) and (ii).

We consider $\text{Com } D = (e_{ij})_{i,j}$ the comatrix of D . Since the action of G on $S(V^*)$ is compatible with the algebra structure, we have

$$g \cdot \text{Com } D = \text{Com } (g \cdot D) = \text{Com } (D g_{M^*}) = \text{Com } D \text{Com } g_{M^*} = \det(g_{M^*}) \text{Com}(D) g_M.$$

We then define, for $j \in \llbracket 1, r \rrbracket$,

$$\mu_j = \sum_{i=1}^r e_{ji} \otimes y_i.$$

We have $\mu_1 \wedge \cdots \wedge \mu_r \in k^\times \det(\text{Com } D) \text{vol}_M$. Although $\det \text{Com } D = (\det D)^{r-1}$ and $\det D \in k^\times Q_{M^*}$ (see [10, proposition 2.4 (iii)]). It remains to show that μ_j is \det_{M^*} -invariant for all $j \in \llbracket 1, r \rrbracket$. But,

$$g e_{ji} = \det g_{M^*} \sum_{k=1}^r e_{jk} g_{M_{ki}} \quad \text{and} \quad g y_i = \sum_{n=1}^r g_{M^*_{ni}} y_n.$$

Hence

$$g \mu_j = \det g_{M^*} \sum_{i=1}^r \left(\sum_{k=1}^r e_{jk} g_{M_{ki}} \otimes \sum_{n=1}^r g_{M^*_{ni}} y_n \right) = \det g_{M^*} \sum_{k=1}^r \sum_{n=1}^r \left(\sum_{i=1}^r g_{M_{ki}} g_{M^*_{ni}} \right) e_{jk} \otimes y_n.$$

Since ${}^t g_{M^*} = g_M^{-1}$, we have $\sum_{i=1}^r g_{M_{ki}} g_{M^*_{ni}} = \delta_{kn}$ and then

$$g \mu_j = \det g_{M^*} \sum_{k=1}^r e_{jk} \otimes y_k = \det g_{M^*} \mu_j.$$

Finally μ_j is $\det_{g_{M^*}}$ -invariant. ■

The following lemma proposes polynomial relations : the aim (with an eye to proposition 24) is to obtain formulas to determine when $Q_\chi Q_M (Q_{\chi \cdot \det_M})^{-1}$ and $Q_{\chi \cdot \det_M} Q_{M^*} (Q_\chi)^{-1}$ are prime to each other.

Lemma 20 – Polynomial identities. We define

$$\mathcal{G}_0 = \{H \in \mathcal{G}, \quad n_H(M) = 0\}, \quad \mathcal{G}_+ = \{H \in \mathcal{G}, \quad n_H(M) \geq e_H - n_H(\chi)\},$$

$$\mathcal{G}_{\neq 0} = \mathcal{G} \setminus \mathcal{G}_0 \quad \text{and} \quad \mathcal{G}_- = \mathcal{G} \setminus \mathcal{G}_+.$$

We have

(i)

$$Q_{M^*} = \prod_{H \in \mathcal{B}} \alpha_H^{n_H(M^*)} \prod_{H \in \mathcal{G}_{\neq 0}} \alpha_H^{e_H(r - n_{0,H}(M)) - n_H(M)}.$$

(ii)

$$Q_{\chi \cdot \det_M} = \prod_{H \in \mathcal{B}} \alpha_H^{n_H(\chi \cdot \det_M)} \prod_{H \in \mathcal{G}_-} \alpha_H^{n_H(\chi) + n_H(M)} \prod_{H \in \mathcal{G}_+} \alpha_H^{n_H(\chi) + n_H(M) - e_H}.$$

(iii)

$$Q_\chi Q_M (Q_{\chi \cdot \det_M})^{-1} = \prod_{H \in \mathcal{B}} \alpha_H^{n_H(\chi) + n_H(M) - n_H(\chi \cdot \det_M)} \prod_{H \in \mathcal{G}_+} \alpha_H^{e_H}.$$

(iv) $Q_{\chi \cdot \det_M} Q_{M^*} (Q_\chi)^{-1} =$

$$\prod_{H \in \mathcal{B}} \alpha_H^{n_H(M^*) + n_H(\chi \cdot \det_M) - n_H(\chi)} \prod_{H \in \mathcal{G}_- \setminus \mathcal{G}_0} \alpha_H^{e_H(r - n_{0,H}(M))} \prod_{H \in \mathcal{G}_+} \alpha_H^{e_H(r - 1 - n_{0,H}(M))}.$$

Proof. We have seen in the remark 9 that $n_H(M^*) = e_H(r - n_{0,H}(M)) - n_H(M)$ for every $H \in \mathcal{H}$. Moreover, $n_H(M) = 0$ if and only if $n_H(M^*) = 0$ if and only if $n_{0,H}(M) = r$. We then obtain (i).

Let $H \in \mathcal{G}$. We have $0 \leq n_H(M) \leq e_H - 1$ and then $n_H(M) = n_H(\det_M)$. We conclude that $(\chi \cdot \det_M)(s_H) = \det(s_H)^{-n_H(\chi) - n_H(M)}$. Since $0 \leq n_H(\chi) + n_H(M) \leq 2e_H - 2$, we obtain $n_H(\chi \cdot \det_M) = n_H(\chi) + n_H(M)$ if $n_H(\chi) + n_H(M) \leq e_H - 1$ and

$$n_H(\chi \cdot \det_M) = n_H(\chi) + n_H(M) - e_H \quad \text{if} \quad n_H(\chi) + n_H(M) \geq e_H.$$

Identities (iii) and (iv) are easy consequences of (i) and (ii). ■

So that we can conclude on the algebra structure of Ω^χ , we need to reinforce hypothesis.

Hypothesis 21 The subset \mathcal{B} contains all the hyperplanes that are not (M, χ) -good that is to say every element of \mathcal{G} that are (M, χ) -good or equivalently $\mathcal{G}_+ = \emptyset$.

Hypothesis 22 The subset \mathcal{B} contains all hyperplanes that are not M -excellents, or equivalently that s_H acts on M as identity or as a reflection for all $H \in \mathcal{G}$.

Remark 23 – Links between hypotheses. The remark 9 ensures that both hypotheses 21 and 22 are stronger than hypothesis 14. In addition, under hypothesis 21, the lemma 20 shows that $Q_\chi Q_M (Q_{\chi \cdot \det_M})^{-1}$ are invertible in $(T^G)^{-1}R$. ■

Proposition 24 – Checking of the necessary and sufficient condition. Let us assume that one of the two hypotheses 21 or 22 are verified. If $\omega_1, \dots, \omega_r$ generate $(\Omega^1)^\chi$ then

$$\omega_1 \wedge \dots \wedge \omega_r \doteq Q_{\chi \cdot \det_M} \text{vol}_M.$$

Proof. From remark 23, the hypothesis 14 is verified. Thus we can define the algebra structure on $(T^{-1}S(V^*) \otimes \Lambda(M^*))^\chi$. Since Ω^χ is stable by \wedge , we have $\omega_1 \wedge \cdots \wedge \omega_r \in (\Omega^r)^\chi$. The identity (1) tells us

$$\exists f \in (T^G)^{-1}R, \quad \omega_1 \wedge \cdots \wedge \omega_r = f Q_{\chi \cdot \det_M} \text{vol}_M.$$

Now, we have to prove that f is invertible in $(T^{-1}S(V^*))^G$. But, actually it suffice to show that f is invertible in $T^{-1}S(V^*)$. Let us consider $(y_i)_{1 \leq i \leq r}$ a basis of M^* . We denote by $C \in M_r(T^{-1}S(V^*))$ the matrix of the family $(\omega_i)_{1 \leq i \leq r}$ in the $T^{-1}S(V^*)$ -basis $(1 \otimes y_i)_{1 \leq i \leq r}$ of Ω^1 . We deduce the existence of $\lambda \in k^\times$ so that

$$\omega_1 \wedge \cdots \wedge \omega_r = \lambda \det C \text{vol}_M \quad \text{then} \quad \lambda \det C = f Q_{\chi \cdot \det_M} (Q_\chi)^{r-1}.$$

In addition, since ν_i is G -invariant, $Q_\chi \nu_i$ is χ -invariant. We then deduce that $Q_\chi \nu_i$ is a linear combination (with coefficients in $(T^G)^{-1}R$) of the family $(\omega_i)_{1 \leq i \leq r}$. Thus we obtain a matrix $D \in M_r((T^G)^{-1}R)$ so that

$$Q_\chi \nu_1 \wedge \cdots \wedge Q_\chi \nu_r = \det D \omega_1 \wedge \cdots \wedge \omega_r = \lambda \det D \det C \text{vol}_M = \lambda \det D f Q_{\chi \cdot \det_M} (Q_\chi)^{r-1} \text{vol}_M$$

But $\nu_1 \wedge \cdots \wedge \nu_r \in k^\times Q_M \text{vol}_M$, so we obtain $Q_\chi \nu_1 \wedge \cdots \wedge Q_\chi \nu_r \in k^\times (Q_\chi)^r Q_M \text{vol}_M$. Hence

$$Q_\chi Q_M \in k^\times f Q_{\chi \cdot \det_M} \det D.$$

Finally $f \neq 0$ and $f \mid Q_\chi Q_M (Q_{\chi \cdot \det_M})^{-1}$.

Under hypothesis 21, the remark 23 shows that f is invertible and (2) is verified.

Now, let us assume hypothesis 22. For $H \in \mathcal{G}_+$, we have $n_{0,H}(M) = r - 1$. Lemma 20 shows that $(Q_{\chi \cdot \det_M} Q_{M^*} (Q_\chi)^{-1})^{r-1}$ and $Q_\chi Q_M (Q_{\chi \cdot \det_M})^{-1}$ are prime to each other. So that, we can conclude by showing that $f \mid (Q_{\chi \cdot \det_M} Q_{M^*} (Q_\chi)^{-1})^{r-1}$.

Let $(\mu_i)_{1 \leq i \leq r}$ be the family of lemma 19. Since μ_i is \det_{M^*} -invariant, $Q_{\chi \cdot \det_M} \mu_i \in (\Omega^1)^\chi$. Then $Q_{\chi \cdot \det_M} \mu_i$ is a linear combination (with coefficients in $(T^G)^{-1}R$) of $(\omega_i)_{1 \leq i \leq r}$. By this way, we have constructed a matrix $D' \in M_r((T^G)^{-1}R)$ so that

$$Q_{\chi \cdot \det_M} \mu_1 \wedge \cdots \wedge Q_{\chi \cdot \det_M} \mu_r = \det D' \omega_1 \wedge \cdots \wedge \omega_r = \det D' f Q_{\chi \cdot \det_M} (Q_\chi)^{r-1} \text{vol}_M.$$

But $\mu_1 \wedge \cdots \wedge \mu_r \in k^\times (Q_{M^*})^{r-1} \text{vol}_M$, so we obtain

$$Q_{\chi \cdot \det_M} \mu_1 \wedge \cdots \wedge Q_{\chi \cdot \det_M} \mu_r \in k^\times (Q_{\chi \cdot \det_M})^r (Q_{M^*})^{r-1} \text{vol}_M.$$

Hence $(Q_{\chi \cdot \det_M} Q_{M^*})^{r-1} \in k^\times \det D' f (Q_\chi)^{r-1}$. Finally $f \mid (Q_{\chi \cdot \det_M} Q_{M^*} (Q_\chi)^{-1})^{r-1}$. Thus, f divides $Q_\chi Q_{\det_M} (Q_{\chi \cdot \det_M})^{-1}$ and $(Q_{\chi \cdot \det_M} Q_{M^*} (Q_\chi)^{-1})^{r-1}$ and identity (2) is verified under hypothesis 22. \blacksquare

Theorem 25 – Exterior Algebra. Let us assume that one of the two hypotheses 21 or 22 are verified. The $(T^G)^{-1}R$ -algebra (Ω^χ, \wedge) is an exterior algebra.

Proof. From propositions 18 and 24 and the remark 23, it suffices to show that $(\Omega^1)^\chi$ can be generated by r elements. Actually, we will show that $(\Omega^1)^\chi$ is a free module of rank r over $(T^G)^{-1}R$. By theorem B of Chevalley [4], we have

$$(S(V^*) \otimes M^*)^\chi = (S(V^*) \otimes M^* \otimes k_\chi^*)^G \otimes k_\chi = (S(V^*) \otimes (M \otimes k_\chi)^*)^G \otimes k_\chi.$$

Thus we obtain

$$(S(V^*) \otimes M^*)^\chi = R \otimes (S_G \otimes (M \otimes k_\chi)^*)^G \otimes k_\chi$$

and $(S(V^*) \otimes M^*)^\chi$ is a free module of rank $\dim_k(\text{Hom}_G(S_G, M \otimes k_\chi)) = \dim_k(M \otimes k_\chi) = r$. By extending the scalar to $(T^G)^{-1}R$, we obtain that $(\Omega^1)^\chi$ is free of rank r . ■

Remark 26 – Shepler, Orlik and Solomon. If every hyperplane of \mathcal{H} is (M, χ) -good, we can choose $\mathcal{B} = \emptyset$. Similarly, if s_H acts on M as a reflection or as identity for all $H \in \mathcal{H}$, we can choose $\mathcal{B} = \emptyset$ and thus $T^{-1}S(V^*) = S(V^*)$. We obtain the results of [1] back and thus those of [11] and [14]. ■

Remark 27 – When $\mathcal{B} = \mathcal{H}$. When $\mathcal{B} = \mathcal{H}$, the hypotheses 21 and 22 are verified. Thus Ω^χ is an $(T^G)^{-1}R$ -exterior algebra. ■

3 Consequences of the exterior algebra structure

In this section, we take an interest in the consequences of the structure of Ω^χ when \mathcal{B} is empty. The first of these consequences is an equality between rational functions (corollary 33) generalizing the one of Orlik and Solomon [11, equality 3.7]. In the subsection 3.3, we give various polynomial identities generalizing those of [11], [8], [7] and [3]. These identities leads to characterizations of the regularity of integers.

Hypothesis 28 In this section, we assume that $\mathcal{B} = \emptyset$. Equivalently, we suppose that every hyperplane in \mathcal{H} is (M, χ) -good or that s_H acts on M trivially or as a reflection for all $H \in \mathcal{H}$.

Thus $(S(V^*) \otimes \Lambda(M^*))^\chi$ is an $S(V^*)^G$ -exterior algebra.

3.1 Introduction and notations

In this subsection, we introduce the objects studied next, in particular we set γ an element of the normalizer of G in $\text{GL}(V)$. In addition, since the product of Ω^χ is a deformation of the usual product, we define a new degree which considers the deformation by Q_χ so that we obtain a bigraduation compatible with the algebra structure.

Notation 29 – Bigraduation. Let us consider $S_n \subset S(V^*)$ the vector space of homogeneous polynomial functions with degree n . For $p \in \llbracket 0, r \rrbracket$, we set $\Omega^p = S(V^*) \otimes \Lambda^p(M^*)$ and $\Omega_n^p = S_n \otimes \Lambda^p(M^*)$. Thus, we have

$$\Omega^\chi = \bigoplus_{p=0}^r (\Omega^p)^\chi \quad \text{and} \quad \Omega^\chi = \bigoplus_{n \in \mathbb{N}} \bigoplus_{p=0}^r (\Omega_n^p)^\chi.$$

For $\omega \in (\Omega_n^p)^\chi$, we set $\deg(\omega) = (n, p)$ and $\deg'(\omega) = (n - \deg Q_\chi, p)$. If $\mu \in (\Omega_{n'}^{p'})^\chi$ then

$$\omega \wedge \mu \in (\Omega_{n+n'}^{p+p'}^\chi)^{\deg Q_\chi}$$

and $\deg'(\omega) + \deg'(\mu) = \deg'(\omega \wedge \mu)$.

Definition 30 – Fake degree, exponents and degrees. Let us remind what are the exponents of a representation of a reflection group. We denote by S_G the coinvariant ring of G that is to say the quotient ring of $S(V^*)$ by the ideal generated by the polynomial invariant functions vanishing at the origin. This is a graded G -module which is isomorphic to the regular representation (this is Chevalley's theorem [4]). We denote by $(S_G)_i$ the graded component of S_G of degree i and then define the fake degree of M to be the polynomial

$$F_M(T) = \sum_{i \in \mathbb{N}} \langle (S_G)_i, M \rangle_G T^i \in \mathbb{Z}[T].$$

Since $\langle (S_G)_i, M \rangle_G$ are non negative integers, we can write $F_M(T) = T^{m_1(M)} + \dots + T^{m_r(M)}$. The integers $m_1(M), \dots, m_r(M)$ are called the M -exponents.

By a theorem of Shephard and Todd [13], the ring $S(V^*)^G$ is generated by a family (f_1, \dots, f_ℓ) of ℓ homogeneous algebraically free generators. We denote by $d_i = \deg f_i$. The multiset (d_1, \dots, d_r) is well determined and called the set of invariant degrees of G .

The following lemma will be useful to extend the character χ of the group $\langle G, \gamma \rangle$.

Lemma 31 – Extension. Let M, N and P be three abelian groups and $\varphi: M \rightarrow N, \theta: M \rightarrow P$ be two morphisms of groups. We assume that P is a divisible group. If $\ker \varphi \subset \ker \theta$, there exists a morphism of groups $\tilde{\theta}: N \rightarrow P$ so that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \downarrow \theta & \searrow \tilde{\theta} & \\ P & & \end{array}$$

Proof. Since $\ker \varphi \subset \ker \theta$, we can define a group homomorphism $\theta_1: M/\ker \varphi \simeq \text{Im } \varphi \rightarrow P$ so that $\theta_1 \circ \varphi = \theta$. Since P is divisible, we can extend θ_1 in $\tilde{\theta}: N \rightarrow P$. Finally we obtain $\tilde{\theta} \circ \varphi = \theta_1 \circ \varphi = \theta$. ■

Let us introduce some notations and consider the normalizer \mathcal{N} of G in $\text{GL}(V)$. We choose a semisimple element $\gamma \in \mathcal{N}$ (see [3]). We assume that M is a $\langle G, \gamma \rangle$ -module and that γ acts semisimply on M . Furthermore we assume that the derived group D of $\langle G, \gamma \rangle$ verifies $D \subset \ker \chi$. By applying lemma 31 with $M = G/D(G)$, $N = \langle G, \gamma \rangle/D$, $P = \mathbb{U}$ the groups of complex numbers with module 1 and $\theta = \chi$, we extend χ in a linear character of $\langle G, \gamma \rangle$ (also denoted by χ).

$$\begin{array}{ccc} G & \longrightarrow & \langle G, \gamma \rangle \\ \downarrow & & \downarrow \\ G/D(G) & \xrightarrow{\varphi} & \langle G, \gamma \rangle/D \\ \downarrow \chi & \searrow \chi & \\ \mathbb{U} & & \end{array}$$

We denote by k_χ the representation (of $\langle G, \gamma \rangle$) with character χ over k and we define $M_\chi = M \otimes k_\chi$. So M_χ is an $\langle G, \gamma \rangle$ -module and, thanks to theorem B of Chevalley [4], we obtain an isomorphism of graded G -modules and of R -modules

$$(\Omega^1)^\chi = (S(V^*) \otimes M^*)^\chi = (S(V^*) \otimes M^* \otimes k_\chi^*)^G \otimes k_\chi = R \otimes (S_G \otimes M_\chi^*)^G \otimes k_\chi.$$

Thus by definition of the M_χ -exponents, we can choose an R -basis $\mathcal{G} = (\omega_1, \dots, \omega_r)$ of $(\Omega^1)^\chi$ bihomogeneous with degree $\deg'(\omega_i) = (m_i(M_\chi) - \deg(Q_\chi), 1)$. Moreover, the hypothesis $D \subset \ker \chi$ ensure that γ stabilizes the vector space N^χ of χ -invariants of N , for all $\langle G, \gamma \rangle$ -module N . Thus we obtain the isomorphism of graded $\langle \gamma \rangle$ -modules and of R -modules

$$(\Omega^1)^\chi = (S(V^*) \otimes M^*)^\chi = (R \otimes S_G \otimes M^*)^\chi = R \otimes (S_G \otimes M^*)^\chi.$$

Finally, we can assume that the ω_i are eigenvectors for γ . We denote by $\varepsilon_{i,\gamma,\chi}(M)$ the eigenvalue of γ associated to ω_i . Both isomorphisms given above show that the multiset $(\varepsilon_{i,\gamma,\chi}(M), m_i(M_\chi))_i$ does not depend of the choice of the basis of $(\Omega^1)^\chi$.

Remark 32 – m_i, ε_i . When $\chi = 1$ is the trivial character, we set $\varepsilon_{i,\gamma}(M) := \varepsilon_{i,\gamma,1}(M)$. Similarly, when $\gamma = \text{id}$, we set $\varepsilon_{i,\chi}(M) := \varepsilon_{i,\text{id},\chi}(M)$. The family of $\varepsilon_{i,\gamma,\chi}(M)$ depends on the choice of the extension of χ to $\langle G, \gamma \rangle$.

The family $\varepsilon_{i,\gamma}(V)$ can also be considered as the family of eigenvalues of γ so that the associated eigenvectors (P_1, \dots, P_ℓ) are a family of homogeneous and algebraically free generators of R (see [3]). ■

3.2 Rational Functions

Here we follow the ideas of theorem 2.1 and equality 2.3 of [7] and of the proposition 2.3 of [8].

Corollary 33 – Rational functions. If s_H acts on M as the identity or as a reflection for all $H \in \mathcal{H}$ or if $n_H(M) < e_H - n_H(\chi)$ for all $H \in \mathcal{H}$ then

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \frac{\det(1+(g\gamma)_M Y)}{\det(1-g\gamma X)} = X^{\deg(Q_\chi)} \frac{\prod_{i=1}^r (1+\varepsilon_{i,\gamma,\chi}(M) Y X^{m_i(M_\chi)-\deg(Q_\chi)})}{\prod_{i=1}^\ell (1-\varepsilon_{i,\gamma}(V) X^{d_i})}.$$

Proof. The hypothesis $D \subset \ker \chi$ ensure that γ stabilizes N^χ the vector space of χ -invariants of N , for all $\langle G, \gamma \rangle$ -module N . In particular, γ defines a bigraded endomorphism of Ω^χ . In order to show the equality, we compute the graded trace $P_{\Omega^\chi, \gamma}(X, Y)$ of the endomorphism γ of Ω^χ in two different ways. By definition,

$$P_{\Omega^\chi, \gamma}(X, Y) = \sum_{n \in \mathbb{N}} \sum_{p=0}^r \text{tr}(\gamma_{\Omega_n^p}^\chi) X^n Y^p.$$

Since $(\Omega_n^p)^\chi$ is the χ -isotypic component of Ω_n^p ,

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} g_{\Omega_n^p}$$

is a projector on $(\Omega_n^p)^\chi$. Hence

$$\text{tr}(\gamma_{\Omega_n^p}^\chi) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \text{tr}((g\gamma)_{\Omega_n^p}).$$

Thus

$$P_{\Omega^\chi, \gamma}(X, Y) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \sum_{n \in \mathbb{N}} \sum_{p=0}^r \text{tr}((g\gamma)_{\Omega_n^p}) X^n Y^p.$$

Finally, Molien's formulas give us

$$P_{\Omega^\chi, \gamma}(X, Y) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \frac{\det(1+(g\gamma)_M Y)}{\det(1-g\gamma X)}. \quad (3)$$

In addition, propositions 18 and 24 show that $\Omega^\chi = R \otimes \lambda((\Omega^1)^\chi)$ where $\lambda((\Omega^1)^\chi)$ is the k -algebra (for λ) generated by $(\Omega^1)^\chi$. Since the product λ is compatible with \deg' and since the degree of the unit element $e = Q_\chi$ for λ is $\deg'(e) = (0, 0)$, we obtain

$$P_{\lambda((\Omega^1)^\chi), \gamma}(X, Y) = X^{\deg(Q_\chi)} \prod_{i=1}^r (1 + \varepsilon_{i,\gamma,\chi}(M) Y X^{m_i(M_\chi)-\deg(Q_\chi)}).$$

Moreover,

$$P_{R, \gamma}(X) = \prod_{i=1}^\ell (1 - \varepsilon_{i,\gamma}(V) X^{d_i})^{-1},$$

Hence

$$P_{\Omega^\chi, \gamma}(X, Y) = X^{\deg(Q_\chi)} \frac{\prod_{i=1}^r (1 + \varepsilon_{i, \gamma, \chi}(M) Y X^{m_i(M_\chi) - \deg(Q_\chi)})}{\prod_{i=1}^{\ell} (1 - \varepsilon_{i, \gamma}(V) X^{d_i})}. \quad (4)$$

The equalities 3 and 4 give the result. \blacksquare

Remark 34 — $m_i(M_\chi)$. If $n_H(M) < e_H - n_H(\chi)$ for all $H \in \mathcal{H}$, then the multisets

$$\{m_1(M) + \deg(Q_\chi), \dots, m_r(M) + \deg(Q_\chi)\} \quad \text{and} \quad \{m_1(M_\chi), \dots, m_r(M_\chi)\}$$

are the same. Indeed, by following the proof of proposition 24, we notice that, under our hypothesis, the family $(Q_\chi \nu_i)_{1 \leq i \leq r}$ is a basis of $(\Omega^1)^\chi$. But the properties of minimal matrices allow us to choose ν_i bihomogeneous with degree $(m_i(M), 1)$. \blacksquare

3.3 Regular Integers

Similarly to the article of Lehrer and Michel [8] and the article of Lehrer [7], let us see apply identity 33 to the representations V^σ and $V^{*\sigma}$ where $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $d \in \mathbb{N}$ and ξ be a primitive d^{th} root of unity; we then define

$$A_\gamma(d) = \{i \in \llbracket 1, \ell \rrbracket, \quad \varepsilon_{i, \gamma}(V) \xi^{-d_i} = 1\} \quad \text{and} \quad a_\gamma(d) = |A_\gamma(d)|,$$

and for $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$r_i(\sigma, \chi) = \deg(Q_\chi) - m_i(V^\sigma_\chi), \quad r_i^*(\sigma, \chi) = \deg(Q_\chi) - m_i(V^{*\sigma}_\chi),$$

$$B_{\sigma, \gamma}(d, \chi) = \{j \in \llbracket 1, \ell \rrbracket, \quad \varepsilon_{j, \gamma, \chi}(V^\sigma) \xi^{-\sigma} \xi^{r_j(\sigma, \chi)} = 1\} \quad \text{and} \quad b_{\sigma, \gamma}(d, \chi) = |B_{\sigma, \gamma}(d, \chi)|;$$

and

$$B_{\sigma, \gamma}^*(d, \chi) = \{j \in \llbracket 1, \ell \rrbracket, \quad \varepsilon_{j, \gamma, \chi}(V^{*\sigma}) \xi^\sigma \xi^{r_j^*(\sigma, \chi)} = 1\} \quad \text{and} \quad b_{\sigma, \gamma}^*(d, \chi) = |B_{\sigma, \gamma}^*(d, \chi)|.$$

For $h \in \text{End}_{\mathbb{C}}(V)$, we denote by $\det'(h)$ the product of non-zero eigenvalues of h , we denote also by $V(h, \xi) = \ker(h - \xi \text{id})$ the eigenspace of h associated to the eigenvalue ξ and we set $d(h, \xi) = \dim(V(h, \xi))$.

Theorem 35 We have $a_\gamma(d) \leq b_{\sigma, \gamma}(d, \chi)$ and the following identity in $\mathbb{C}[T]$

$$\xi^{\deg(Q_\chi)} \sum_{g \in G} \overline{\chi(g)} T^{d(g, \xi)} (\det'(1 - \xi^{-1} g \gamma))^{\sigma-1} = \begin{cases} \prod_{j \in B_{\sigma, \gamma}(d, \chi)} (T - r_j(\sigma, \chi)) \prod_{j \notin B_{\sigma, \gamma}(d, \chi)} (1 - \varepsilon_j \xi^{r_j(\sigma, \chi) - \sigma}) \prod_{j \notin A_\gamma(d)} \frac{d_j}{1 - \varepsilon'_j \xi^{-d_j}} & \text{if } a_\gamma(d) = b_{\sigma, \gamma}(d, \chi), \\ 0 & \text{otherwise} \end{cases}$$

where $\varepsilon_i = \varepsilon_{i, \gamma, \chi}(V^\sigma)$ and $\varepsilon'_i = \varepsilon_{i, \gamma}(V)$.

We have $a_\gamma(d) \leq b_{\sigma, \gamma}^*(d, \chi)$ and the following identity in $\mathbb{C}[T]$

$$(-1)^\ell \xi^{\deg(Q_\chi) + \ell \sigma} \sum_{g \in G} \overline{\chi(g)} (-T)^{d(g, \xi)} (\det'(1 - \xi^{-1} g \gamma))^{\sigma-1} \det(g \gamma)^{-\sigma} =$$

$$\begin{cases} \prod_{j \in B_{\sigma, \gamma}^*(d, \chi)} (T - r_j^*(\sigma, \chi)) \prod_{j \notin B_{\sigma, \gamma}^*(d, \chi)} (1 - \varepsilon_j \xi^{r_j^*(\sigma, \chi) + \sigma}) \prod_{j \notin A_\gamma(d)} \frac{d_j}{1 - \varepsilon'_j \xi^{-d_j}} & \text{if } a_\gamma(d) = b_{\sigma, \gamma}^*(d, \chi), \\ 0 & \text{otherwise} \end{cases}$$

where $\varepsilon_i = \varepsilon_{i, \gamma, \chi}(V^{*\sigma})$ and $\varepsilon'_i = \varepsilon_{i, \gamma}(V)$.

Proof. For every reflection $s \in G$, s_V^σ is still a reflection. Thus we can apply corollary 33 to the G -module V^σ . For $g \in G$, we denote by $\lambda_1(g\gamma), \dots, \lambda_\ell(g\gamma)$ the eigenvalues of $g\gamma$ acting on V . We have

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \prod_{i=1}^{\ell} \frac{(1+Y(\lambda_i(g\gamma))^\sigma)}{(1-X\lambda_i(g\gamma))} = X^{\deg(Q_\chi)} \prod_{i=1}^{\ell} \frac{(1+\varepsilon_i Y X^{-r_i(\sigma, \chi)})}{(1-\varepsilon'_i X^{d_i})}.$$

We switch the indeterminate with $Y = \xi^{-\sigma}(T(1 - \xi X) - 1)$.

Let us begin with the left side. It becomes

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \prod_{i=1}^{\ell} \frac{1 - (\lambda_i(g\gamma)\xi^{-1})^\sigma (1 - T(1 - \xi X))}{1 - X\lambda_i(g\gamma)}.$$

In each term of the sum, we discriminate the eigenvalues of $g\gamma$ between those equal to ξ and the others. We obtain in $\mathbb{C}(T, X)$

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \left(\prod_{\{i \mid \lambda_i = \xi\}} T \prod_{\{i \mid \lambda_i \neq \xi\}} \frac{1 - (\lambda_i(g\gamma)\xi^{-1})^\sigma (1 - T(1 - \xi X))}{1 - X\lambda_i(g\gamma)} \right).$$

So ξ^{-1} is not a pole of this rational function with respect to X and evaluating at $X = \xi^{-1}$, we obtain

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} T^{d(g\gamma, \xi)} \left(\prod_{\{i \mid \lambda_i \neq \xi\}} \frac{1 - (\xi^{-1}\lambda_i(g\gamma))^\sigma}{1 - \xi^{-1}\lambda_i(g\gamma)} \right) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} T^{d(g\gamma, \xi)} (\det'(1 - \xi^{-1}g\gamma))^{\sigma-1}.$$

Now, let us consider the right side. After switching the indeterminate, it becomes

$$X^{\deg(Q_\chi)} \prod_{i=1}^{\ell} \frac{1 - \varepsilon_i \xi^{-\sigma} (1 - T(1 - \xi X)) X^{-r_i(\sigma, \chi)}}{1 - \varepsilon'_i X^{d_i}}.$$

Let us count the multiplicity of ξ^{-1} as a root of the numerator and of the denominator of this rational function with respect to X . For the denominator, ξ^{-1} is a root of $1 - \varepsilon'_i X^{d_i}$ if and only if $i \in A_\gamma(d)$. Moreover this root is simple. So ξ^{-1} is a root of order $a_\gamma(d)$ of the denominator. For the numerator,

$$1 - \varepsilon_i \xi^{-\sigma} (1 - T(1 - \xi X)) X^{-r_i(\sigma, \chi)}$$

is zero for $X = \xi^{-1}$ if and only if $i \in B_\sigma(d, \chi)$. Moreover, when differentiating with respect to X , we obtain

$$-\varepsilon_i \xi^{-\sigma} \left(-r_i(\sigma, \chi) (1 - T(1 - \xi X)) X^{-r_i(\sigma, \chi)-1} + T \xi X^{-r_i(\sigma, \chi)} \right)$$

which is nonzero at $X = \xi^{-1}$. So ξ^{-1} is a root of order $b_{\sigma, \gamma}(d, \chi)$ of the numerator.

Since ξ^{-1} is not a pole of the left side, we obtain $a_\gamma(d) \leq b_{\sigma, \gamma}(d, \chi)$. Moreover, we deduce that the right side is zero if $a_\gamma(d) < b_{\sigma, \gamma}(d, \chi)$.

Now, let us assume that $a_\gamma(d) = b_{\sigma, \gamma}(d, \chi)$. If $i \in B_{\sigma, \gamma}(d, \chi)$ then $\varepsilon_i \xi^{-\sigma} = \xi^{-r_i(\sigma, \chi)}$ and

$$\begin{aligned} 1 - \varepsilon_i \xi^{-\sigma} (1 - T(1 - \xi X)) X^{-r_i(\sigma, \chi)} &= 1 - (1 - T(1 - \xi X)) (\xi X)^{-r_i(\sigma, \chi)} \\ &= 1 - (\xi X)^{-r_i(\sigma, \chi)} + T(1 - \xi X) (\xi X)^{-r_i(\sigma, \chi)} \\ &= (1 - \xi X) \left(T(\xi X)^{-r_i(\sigma, \chi)} + \sum_{k=0}^{-r_i(\sigma, \chi)-1} (\xi X)^k \right). \end{aligned}$$

For $j \in A_\gamma(d)$, we have $\varepsilon'_j = \xi^{d_j}$ and so

$$\begin{aligned} 1 - \varepsilon'_j X^{d_j} &= 1 - \xi^{d_j} X^{d_j} \\ &= (1 - \xi X) \sum_{k=0}^{d_j-1} (\xi X)^k. \end{aligned}$$

As a consequence, for $j \in A_\gamma(d)$ and $i \in B_{\sigma,\gamma}(d, \chi)$, we obtain

$$\frac{1 - \varepsilon_i \xi^{-\sigma} (1 - T(1 - \xi X)) X^{-r_i(\sigma, \chi)}}{1 - X^{d_j}} = \frac{T(\xi X)^{-r_i(\sigma, \chi)} + \sum_{k=0}^{-r_i(\sigma, \chi)-1} (\xi X)^k}{\sum_{k=0}^{d_j-1} (\xi X)^k}.$$

Evaluating at $X = \xi^{-1}$, we obtain $\frac{T - r_i(\sigma, \chi)}{d_j}$. Finally, by choosing for each factor of the numerator whose index is in $B_{\sigma,\gamma}(d, \chi)$, one of the factor of the denominator whose index is in $A_\gamma(d)$ (this is possible since $a_\gamma(d) = b_{\sigma,\gamma}(d, \chi)$), we obtain, after evaluating at $X = \xi^{-1}$,

$$\xi^{-\deg(Q_\chi)} \frac{\prod_{j \in B_{\sigma,\gamma}(d, \chi)} (T - r_j(\sigma, \chi)) \prod_{j \notin B_{\sigma,\gamma}(d, \chi)} (1 - \varepsilon_j \xi^{r_j(\sigma, \chi) - \sigma})}{\prod_{j \in A_\gamma(d)} d_j \prod_{j \notin A_\gamma(d)} (1 - \varepsilon'_j \xi^{-d_j})}.$$

The relation $|G| = \prod_{i=1}^{\ell} d_i$ give us the identity.

For the second identity, we apply the corollary 33 to $V^{*\sigma} = V^{\sigma*}$ on which s_H acts as a reflection for all $H \in \mathcal{H}$. We switch the indeterminate in $Y = \xi^\sigma (T(1 - \xi X) - 1)$ and simplify with $(1 - z^{-1})(1 - z)^{-1} = -z^{-1}$. ■

3.3.1 When γ is trivial

We are interested in the case where $\gamma = \text{id}$. To simplify the notations, we set

$$B^*(d, \chi) := B_{\text{id}, \text{id}}^*(d, \chi) = \{j \in \llbracket 1, \ell \rrbracket, \quad d \mid 1 + r_j^*(\text{id}, \chi)\} \quad \text{and} \quad b^*(d, \chi) = |B^*(d, \chi)|;$$

$$B(d, \chi) := B_{\text{id}, \text{id}}(d, \chi) = \{j \in \llbracket 1, \ell \rrbracket, \quad d \mid 1 - r_j(\text{id}, \chi)\} \quad \text{and} \quad b(d, \chi) = |B(d, \chi)|;$$

and finally $A(d) := A_{\text{id}}(d) = \{j \in \llbracket 1, \ell \rrbracket, \quad d \mid d_j\}$ and $a(d) = |A(d)|$.

Let us remind that d is said to be a regular integer if one of the $V(g, \xi)$ meets the complementary of the hyperplanes of \mathcal{H} . The following corollary generalizes the results of [8] and the one of [7].

Corollary 36 – Consequences and Exceptional Case. We obtain the following formulas

$$\begin{aligned} (i) \quad & \sum_{g \in G} \overline{\chi(g)} T^{d(g, 1)} (\det'(1 - g))^{\sigma-1} = \prod_{j=1}^{\ell} (T - r_j(\sigma, \chi)). \\ (ii) \quad & \xi^{\deg(Q_\chi)} \sum_{g \in G} \overline{\chi(g)} T^{d(g, \xi)} = \\ & \begin{cases} \prod_{j \in B(d, \chi)} (T - r_j(\text{id}, \chi)) \prod_{j \notin B(d, \chi)} (1 - \xi^{r_j(\text{id}, \chi) - 1}) \prod_{j \notin A(d)} \frac{d_j}{1 - \xi^{-d_j}}, & \text{if } a(d) = b(d, \chi), \\ 0 & \text{otherwise.} \end{cases} \\ (iii) \quad & \sum_{g \in G} \chi(g) T^{d(g, 1)} = \prod_{j=1}^{\ell} (T - r_j(\text{id}, \chi)). \end{aligned}$$

- (iv) We have $a(d) \leq b^*(d, \chi)$ and
- $$(-1)^\ell \xi^{\ell + \deg(Q_\chi)} \sum_{g \in G} (-T)^{d(g, \xi)} (\chi \cdot \det)(g^{-1}) = \begin{cases} \prod_{j \in B^*(d, \chi)} (T - r_j^*(\text{id}, \chi)) \prod_{j \notin B^*(d, \chi)} (1 - \xi^{r_j^*(\text{id}, \chi)+1}) \prod_{j \notin A(d)} \frac{d_j}{1 - \xi^{-d_j}}, & \text{if } a(d) = b^*(d, \chi), \\ 0 & \text{otherwise.} \end{cases}$$
- (v) $\sum_{g \in G} T^{d(g, 1)} (\chi \cdot \det)(g) = \prod_{j=1}^{\ell} (T + r_j^*(\text{id}, \chi)).$
- (vi) The multisets $\{-r_1^*(\text{id}, \chi), \dots, -r_\ell^*(\text{id}, \chi)\}$ and $\{r_1(\text{id}, \chi \cdot \det), \dots, r_\ell(\text{id}, \chi \cdot \det)\}$ are the same and $b^*(d, \chi) = b(d, \chi \cdot \det).$
- (vii) If d is regular, then $a(d) = b_\sigma(d, \chi)$ for every $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and every one dimensional character χ .
- (viii) If for all $H \in \mathcal{H}$, the restriction of $\chi \cdot \det$ to G_H is non trivial, then d is a regular integer if and only if $a(d) = b(d, \chi).$

Proof.

- (i) This is theorem 35 with $d = 1$ and so $\xi = 1$. We have

$$A(1) = B_{\sigma, \text{id}}(1, \chi) = \llbracket 1, \ell \rrbracket \quad \text{and} \quad a(1) = b_{\sigma, \text{id}}(1, \chi).$$

- (ii) This is theorem 35 with $\sigma = \text{id}$. This Lehrer's identity 2.1 [7].
- (iii) Since $\overline{\chi(g)} = \chi(g^{-1})$ and $d(g, 1) = d(g^{-1}, 1)$, this is (i) for $\sigma = \text{id}$ or (ii) with $d = 1$. This is Lehrer's identity 3.2 [7].
- (iv) This is the second identity of theorem 35 for $\sigma = \text{id}$.
- (v) This is (iv) with $d = 1 = \xi$ with the remark that $d(g, 1) = d(g^{-1}, 1)$.
- (vi) This one is obtained by comparing (iii) applied to $\chi \cdot \det$ with (v).
- (vii) Theorem 3.4 of Springer [15] shows us that the degree of the polynomial of the left side in theorem 35 is at most $a(d)$. Let us compute the coefficient of $T^{a(d)}$. Since d is regular, the $g \in G$ verifying $d(g, \xi) = a(d)$ are a single conjugacy class. Thus, for every g so that $d(g, \xi) = a(d)$, the value of $\overline{\chi(g)} \det'(1 - \xi^{-1}g)^{\sigma-1}$ does not depend on g . The coefficient of $T^{a(d)}$ is non zero and so $a(d) = b_\sigma(d, \chi).$
- (viii) This is (vi) coupled with corollary 3.9 of Lehrer [7] applied to the linear character $\chi \cdot \det$. ■

3.3.2 When γ is not necessarily trivial

The case $\chi = 1$ is done in [3]. Let us remind that d is γ -regular if one of the eigenspaces $V(g\gamma, \xi)$ meets the complementary of the hyperplanes \mathcal{H} . As a matter of simplification, we set $B_\gamma^*(d, \chi) := B_{\text{id}, \gamma}^*(d, \chi)$, $b_\gamma^*(d, \chi) := |B_\gamma^*(d, \chi)|$ and

$$B_\gamma(d, \chi) := B_{\text{id}, \gamma}(d, \chi) \quad \text{and} \quad b_\gamma(d, \chi) := |B_\gamma(d, \chi)|.$$

Corollary 37 – Consequences and Exceptionnal Cases. We obtain the following formulas

$$(i) \quad \xi^{\deg(Q_\chi)} \sum_{g \in G} \overline{\chi(g)} T^{d(g, \xi)} = \begin{cases} \prod_{j \in B_\gamma(d, \chi)} (T - r_j(\text{id}, \chi)) \prod_{j \notin B_\gamma(d, \chi)} (1 - \varepsilon_j \xi^{r_j(\text{id}, \chi)-1}) \prod_{j \notin A_\gamma(d)} \frac{d_j}{1 - \varepsilon_j' \xi^{-d_j}}, & \text{if } a_\gamma(d) = b_\gamma(d, \chi), \\ 0 & \text{otherwise.} \end{cases}$$

when $\varepsilon_i = \varepsilon_{i, \gamma, \chi}(V)$ and $\varepsilon_i' = \varepsilon_{i, \gamma}(V).$

- (ii) $(-1)^\ell \xi^{\deg(Q_\chi) + \ell} \det(\gamma^{-1}) \sum_{g \in G} (\chi \cdot \det)(g^{-1}) (-T)^{d(g\gamma, \xi)} =$
- $$\begin{cases} \prod_{j \in B_\gamma^*(d, \chi)} (T - r_j^*(\text{id}, \chi)) \prod_{j \notin B_\gamma^*(d, \chi)} (1 - \varepsilon_j \xi^{r_j^*(\text{id}, \chi) + 1}) \prod_{j \notin A_\gamma(d)} \frac{d_j}{1 - \varepsilon_j' \xi^{-d_j}} & \text{if } a_\gamma(d) = b_\gamma^*(d, \chi), \\ 0 & \text{otherwise} \end{cases}$$
- when $\varepsilon_i = \varepsilon_{i, \gamma, \chi}(V^*)$ and $\varepsilon_i' = \varepsilon_{i, \gamma}(V)$.
- (iii) The two multisets $\{-r_i^*(\text{id}, \chi), i \in B_\gamma^*(d, \chi)\}$ and $\{r_i(\text{id}, \chi \cdot \det), i \in B_\gamma(d, \chi)\}$ are the same and $b_\gamma^*(d, \chi) = b_\gamma(d, \chi \cdot \det)$.
- (iv) If d is γ -regular, then $a_\gamma(d) = b_{\sigma, \gamma}(d, \chi) = b_{\sigma, \gamma}^*(d, \chi)$ for every linear character χ and every $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
- (v) If for all $H \in \mathcal{H}$, the restriction of χ to G_H is non-trivial, then d is γ -regular if and only if $a_\gamma(d) = b_\gamma(d, \chi)$.
- (vi) If for all $H \in \mathcal{H}$, the restriction of $\chi \cdot \det$ to G_H is non-trivial, then d is γ -regular if and only if $a_\gamma(d) = b_\gamma^*(d, \chi)$.

Proof.

- (i) This is theorem 35 with $\sigma = \text{id}$.
- (ii) This is the second identity of theorem 35 with $\sigma = \text{id}$.
- (iii) Let us compare the roots of (i) applied to $\chi \cdot \det$ and those of (ii).
- (iv) The theorem 3.4 of Springer [15] show us that the degree of polynomial of the left side in theorem 35 is at most $a_\gamma(d)$. Let us compute the coefficient of $T^{a_\gamma(d)}$. Since d is regular, the $g \in G$ so that $d(g\gamma, \xi) = a_\gamma(d)$ are a single conjugacy class. Thus the value $\overline{\chi(g)} \det'(1 - \xi^{-1}g)^{\sigma-1}$ does not depend on g when g verifies $d(g\gamma, \xi) = a_\gamma(d)$. The coefficient of $T^{a_\gamma(d)}$ is non-zero and so $a_\gamma(d) = b_{\sigma, \gamma}(d, \chi)$.
- (v) By (iv), it suffices to show that if $a_\gamma(d) = b_\gamma(d, \chi)$ then d is γ -regular. By (i), the coefficient of $T^{a_\gamma(d)} = T^{b_\gamma(d, \chi)}$ is non-zero. Thus, thanks to Springer theorem, $\sum_{g \in C} \chi(g)$ is a factor of the coefficient of $T^{a_\gamma(d)}$ where

$$C = \{g \in G, \quad \forall x \in V(h\gamma, \xi), \quad gx = x\} \quad \text{with} \quad d(h\gamma, \xi) = a_\gamma(d).$$

If C is not the trivial group, then C contains one of the G_H (this is Steinberg's theorem) and since the restriction of χ to G_H is non-trivial, we have $\sum_{g \in C} \chi(g) = 0$. Finally we obtain a contradiction and $C = 1$ which means exactly that d is γ -regular.

- (vi) This is (iii) and (v). ■

4 Types of hyperplanes

In definition 8, we define various types of hyperplanes. In this section, we study these types of hyperplane for some examples of reflection groups, namely the symmetric group, the wreath product $G(d, 1, n)$, the imprimitive groups of rank 2 that is $G(de, e, 2)$ and some exceptional case G_4 , G_5 and G_{24} (named after the classification of [13]). The details of the computations can be found in [2].

4.1 The symmetric group

The symmetric groups \mathfrak{S}_n acts faithfully as a reflection group over $\mathbb{C}^n / \langle (1, \dots, 1) \rangle$ by permuting the coordinates. The reflections are the transpositions. They are of order 2 and conjugate to each other. Hence there is a unique conjugacy class of hyperplane. The linear character of

\mathfrak{S}_n are the trivial one (denoted by 1) and the sign character (denoted by ε). In addition, the representation of \mathfrak{S}_n are described by the partition of n (see for example [9]).

Proposition 38 – Symmetric group. Let H be an hyperplane of the reflection group \mathfrak{S}_n and ρ be an irreducible representation of \mathfrak{S}_n .

The hyperplane H is ρ -excellent, ρ -good, $(\rho, 1)$ -good, (ρ, ε) -acceptable if and only if $\rho = 1$ or $\rho = \varepsilon$ or ρ is the reflection representation or $n = 4$ and ρ is associated to the partition $(2, 2)$.

The hyperplane H is (ρ, ε) -good if and only if $\rho = 1$.

The hyperplane H is (ρ, ε) -acceptable if and only if $\rho = 1$ or $\rho = \varepsilon$ or ρ is the reflection representation or $n \leq 5$ or $n = 6$ and ρ is associated to one of the partition $(3, 3)$, $(2, 2, 2)$, $(4, 2)$.

4.2 The rank 2 imprimitive groups

For d, e, r nonzero integers, we define the group $G(de, e, r)$ to be the group of r -dimensional monomial matrices (with one nonzero element on each row and column) whose nonzero entries are d^{th} root of unity such that their product is a d^{th} root of unity. These groups are called the imprimitive groups of reflection. The integer r is called the rank of $G(de, e, r)$.

The reflections of $G(de, e, 2)$ are of the form

$$\begin{bmatrix} \xi & \\ & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & \\ & \xi \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} & \zeta \\ \zeta^{-1} & \end{bmatrix}$$

where ξ is a non trivial d^{th} root of unity and ζ a de^{th} root of unity. If e is odd, there are two conjugacy classes of hyperplanes : one given by the hyperplanes of the diagonal reflections, the other given by the nondiagonal reflections. If e is even, there are three conjugacy classes of hyperplanes. One given by the hyperplanes of the diagonal reflections. The hyperplanes of nondiagonal reflections split into two classes : one associated to the hyperplane of

$$s = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$$

the other one associated to the hyperplane of

$$s' = \begin{bmatrix} & \zeta \\ \zeta^{-1} & \end{bmatrix}$$

where ζ is a de^{th} primitive root of unity.

We will describe the character of $G(de, e, 2)$ following the method of small groups of Wigner and Mackey (see [12, paragraph 8.2]). So we set D the subgroup of diagonal matrix of $G(de, e, 2)$. This is an abelian normal subgroup of $G(de, e, 2)$ of index 2. To use the method of Wigner and Mackey, we have to describe the one-dimensional character of D .

Lemma 39 – Linear character of D . For $d, e \in \mathbb{N}^*$, the map

$$\Delta: \begin{cases} \mathbb{Z}/de\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \longrightarrow \widehat{D} = \text{Hom}_{gr}(D, \mathbb{C}^\times) \\ (k, k') \longmapsto (\text{diag}(\alpha, \beta) \mapsto \alpha^{-k}(\alpha\beta)^{-k'}) \end{cases}$$

is a group isomorphism.

Let us now describe the irreducible representation of $G(de, e, 2)$. We have to distinguish with the evenness of d and e .

Proposition 40 – d, e odd. Let $d, e \in \mathbb{N}^*$ be odd numbers.

For $k' \in \mathbb{Z}/d\mathbb{Z}$, we extend $\Delta(0, k')$ to $G(de, e, 2)$ by $\Delta(0, k')(dx) = \Delta(0, k')(d)$ for every $d \in D$ and $x \in \langle 1, s \rangle$. We extend any irreducible representation ρ of $\langle 1, s \rangle$ to $G(de, e, 2)$ by $\rho(dx) = \rho(x)$ for every $d \in D$ and $x \in \langle 1, s \rangle$. We then define

$$\beta_{k', \rho}(dx) = \rho(dx)\Delta(0, k')(dx) = \rho(x)\Delta(0, k')(d)$$

for $d \in D$ and $x \in \langle 1, s \rangle$. For $(k, k') \in \llbracket 1, de/2 \rrbracket \times \mathbb{Z}/d\mathbb{Z}$, we set $\beta_{k, k'} = \text{Ind}_D^{G(de, e, 2)} \Delta(k, k')$.

The family $((\beta_{k', 1}, \beta_{k', \varepsilon})_{k' \in \mathbb{Z}/d\mathbb{Z}}, (\beta_{k, k'})_{(k, k') \in \llbracket 1, de/2 \rrbracket \times \mathbb{Z}/d\mathbb{Z}})$ is a complete set for the irreducible representations of $G(de, e, 2)$.

Proposition 41 — e odd, d even. Let $d, e \in \mathbb{N}^*$ with $d = 2d'$ even and e odd.

For $k' \in \mathbb{Z}/d\mathbb{Z}$, we extend $\Delta(0, k')$ to $G(de, e, 2)$ by $\Delta(0, k')(dx) = \Delta(0, k')(d)$ for every $d \in D$ and $x \in \langle 1, s \rangle$. We extend any irreducible representation ρ of $\langle 1, s \rangle$ to $G(de, e, 2)$ by $\rho(dx) = \rho(x)$ for every $d \in D$ and $x \in \langle 1, s \rangle$. We then define

$$\beta_{k', \rho}(dx) = \rho(dx)\Delta(0, k')(dx) = \rho(x)\Delta(0, k')(d)$$

for every $d \in D$ and $x \in \langle 1, s \rangle$.

We denote by A the set $A = \{\llbracket 1, d'e - 1 \rrbracket \times \mathbb{Z}/d\mathbb{Z}\} \cup \{d'e\} \times \llbracket 0, d' - 1 \rrbracket$. For $(k, k') \in A$, we set

$$\beta_{k, k'} = \text{Ind}_D^{G(de, e, 2)} \Delta(k, k').$$

The family $((\beta_{k', 1}, \beta_{k', \varepsilon})_{k' \in \mathbb{Z}/d\mathbb{Z}}, (\beta_{k, k'})_{(k, k') \in A})$ is a complete set of irreducible representations of $G(de, e, 2)$.

Proposition 42 — e even. Let $d, e \in \mathbb{N}^*$ with $e = 2e'$ even.

For $k' \in \mathbb{Z}/d\mathbb{Z}$ and $\delta \in \{0, de'\}$, we extend the character $\Delta(\delta, k')$ to $G(de, e, 2)$ by $\Delta(\delta, k')(dx) = \Delta(\delta, k')(d)$ for $d \in D$ and $x \in \langle 1, s \rangle$. We extend any irreducible representation ρ of $\langle 1, s \rangle$ to $G(de, e, 2)$ by $\rho(dx) = \rho(x)$ for every $d \in D$ and $x \in \langle 1, s \rangle$. We then define, for $d \in D$ and $x \in \langle 1, s \rangle$,

$$\beta_{\delta, k', \rho}(dx) = \rho(dx)\Delta(\delta, k')(dx) = \rho(x)\Delta(\delta, k')(d).$$

For $(k, k') \in \llbracket 1, de' - 1 \rrbracket \times \mathbb{Z}/d\mathbb{Z}$, we set

$$\beta_{k, k'} = \text{Ind}_D^{G(de, e, 2)} \Delta(k, k').$$

The family $((\beta_{\delta, k', 1}, \beta_{\delta, k', \varepsilon})_{(\delta, k') \in \{0, de'\} \times \mathbb{Z}/d\mathbb{Z}}, (\beta_{k, k'})_{(k, k') \in \llbracket 1, de' - 1 \rrbracket \times \mathbb{Z}/d\mathbb{Z}})$ is a complete set of irreducible representations of $G(de, e, 2)$.

Corollary 43 — Non diagonal hyperplanes. Let $d, e \in \mathbb{N}^*$, ρ be an irreducible representation of $G(de, e, 2)$, χ a linear character of $G(de, e, r)$ and H the hyperplane of a non-diagonal reflection.

The hyperplane H is ρ -good, ρ -excellent and (ρ, χ) -acceptable for every ρ and χ .

If e is odd and ρ is a 2-dimensional representation of $G(de, e, 2)$, the hyperplane H is (ρ, χ) -good if and only if $\chi = \beta_{k', 1}$ with $k' \in \mathbb{Z}/d\mathbb{Z}$.

If e is odd, $\rho = \beta_{k', \rho'}$ is a 1-dimensional representation of $G(de, e, 2)$ and $\chi = \beta_{k'', \rho''}$ a linear character of $G(de, e, 2)$, then the hyperplane H is (ρ, χ) -good if and only if $\rho' \neq \varepsilon$ or $\rho'' \neq \varepsilon$.

Assume that $e = 2e'$ is even, H is an hyperplane associated to the conjugacy class of s and ρ a 2-dimensional representation of $G(de, e, 2)$ then H is (ρ, χ) -good if and only if $\chi = \beta_{\delta, k', 1}$ with $\delta \in \{0, de'\}$ and $k' \in \mathbb{Z}/d\mathbb{Z}$.

Assume that $e = 2e'$ is even, H is an hyperplane associated to the conjugacy class of s , $\rho = \beta_{u,k',\rho'}$ is a 1-dimensional representation of $G(de, e, 2)$ and $\chi = \beta_{v,k'',\rho''}$ a linear character of $G(de, e, 2)$, then H is (ρ, χ) -good if and only if $\rho'' \neq \varepsilon$ or $\rho' \neq \varepsilon$.

Assume that $e = 2e'$ is even, H is an hyperplane associated to the conjugacy class of s' and ρ a 2-dimensional representation of $G(de, e, 2)$ then H is (ρ, χ) -good if and only if $\chi = \beta_{0,k',1}$ or $\chi = \beta_{de',k',\varepsilon}$ with $k' \in \mathbb{Z}/d\mathbb{Z}$.

Assume that $e = 2e'$ is even, H is an hyperplane associated to the conjugacy class of s' , $\rho = \beta_{u,k',\rho'}$ is a 1-dimensional representation of $G(de, e, 2)$ and $\chi = \beta_{v,k'',\rho''}$ a linear character of $G(de, e, 2)$, then H is (ρ, χ) -good if and only if $(u, \rho') \notin \{(0, \varepsilon), (de', 1)\}$ or $(v, \rho'') \notin \{(0, \varepsilon), (de', 1)\}$.

Corollary 44 – Diagonal hyperplanes. Let $d, e \in \mathbb{N}^*$ and $\beta_{k,k'}$ be a 2-dimensional irreducible representation of $G(de, e, 2)$, χ a linear character of $G(de, e, r)$ and H an hyperplane associated to a diagonal reflection.

For $n \in \mathbb{Z}$, we denote by \bar{n} the unique integer such that $0 \leq \bar{n} \leq d-1$ and $d \mid (n - \bar{n})$.

With these notations, we obtain $n_H(\beta_{k,k'}) = \overline{k+k'} + k'$. For e odd and $u \in \{1, \varepsilon\}$, we have $n_H(\beta_{k',u}) = k'$. For $e = 2e'$ even, $\delta \in \{0, de'\}$ and $u \in \{1, \varepsilon\}$, we have $n_H(\beta_{\delta,k',u}) = k'$.

If $k' = 0$ then H is $\beta_{k,k'}$ -excellent and so $\beta_{k,k'}$ -good and $(\beta_{k,k'}, \chi)$ -acceptable for every χ . Moreover $n_H(\beta_{k,k'}) = \bar{k}$. Thus, for $\chi = \beta_{\delta,k'',u}$ or $\chi = \beta_{k'',u}$ with $\delta \in \{0, de'\}$ and $u \in \{1, \varepsilon\}$, the hyperplane H is $(\beta_{k,k'}, \chi)$ -good if and only if $\bar{k} + k'' < d$.

Let us assume $k' \neq 0$. For $\chi = \beta_{\delta,k'',u}$ or $\chi = \beta_{k'',u}$ with $\delta \in \{0, de'\}$ and $u \in \{1, \varepsilon\}$, the hyperplane H is

- (i) $\beta_{k,k'}$ -excellent if and only if $\overline{k+k'} = 0$;
- (ii) $\beta_{k,k'}$ -good if and only if $\overline{k+k'} + k' < d$;
- (iii) $(\beta_{k,k'}, \chi)$ -good if and only if $\overline{k+k'} + k' + k'' < d$;
- (iv) $(\beta_{k,k'}, \chi)$ -acceptable if and only if $d - k'' > \overline{k+k'}$ or $d - k'' > k'$.

If e is odd, $\rho = \beta_{k',\rho'}$ is a 1-dimensional representation of $G(de, e, 2)$ and $\chi = \beta_{k'',\rho''}$ a linear character of $G(de, e, 2)$, then H is (ρ, χ) -good if and only if $k' + k'' < d$.

Assume that $e = 2e'$ is even, $\rho = \beta_{u,k',\rho'}$ is a 1-dimensional representation of $G(de, e, 2)$ and $\chi = \beta_{v,k'',\rho''}$ a linear character of $G(de, e, 2)$, then the hyperplane H is (ρ, χ) -good if and only if $k' + k'' < d$.

4.3 The wreath product

Let us now study the imprimitive reflection group $G(d, 1, r)$ which is also the wreath product $\mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_r$. Let us assume $r \geq 3$. The reflections of $G(d, 1, r)$ are of the form

$$\text{diag}(1, \dots, 1, \xi, 1, \dots, 1)$$

with ξ a non trivial d^{th} of unity and of the form

$$\text{diag}(1, \dots, 1, \zeta, 1, \dots, 1, \zeta^{-1}, 1, \dots, 1)\tau_{ij}$$

where τ_{ij} is the transposition matrix swapping i and j and ζ is a d^{th} root of unity. The associated hyperplanes split into two conjugacy class : the diagonal one and the non-diagonal one.

The irreducible character of $G(d, 1, r)$ can be described by the method of Wigner and Mackey with the normal abelian subgroup of diagonal matrices of $G(d, 1, r)$. Thus the representation of $G(d, 1, r)$ are given by the d -multipartitions of r that is to say families of d partitions so that the sum of the length of the partitions are r . One can also describe the

irreducible representations of $G(d, 1, r)$ by giving a family of d integers $\underline{r} = (n_0, \dots, n_{d-1})$ such that $n_0 + \dots + n_{d-1} = r$ and ρ a representation of $\mathfrak{S}_{n_0} \times \dots \times \mathfrak{S}_{n_{d-1}}$. We denote by $\beta_{\underline{r}, \rho}$ the corresponding representation of $G(d, 1, r)$.

Corollary 45 – Non diagonal hyperplanes. Let $d \geq 2$, $r \geq 3$, H be a non diagonal hyperplane of $G(d, 1, r)$ (we denote by G_H the subgroup of G of reflections whose hyperplane is H), $\rho' = \beta_{\underline{r}, \rho}$ be an irreducible representation of $G(d, 1, r)$ and χ a linear character of $G(d, 1, r)$.

- (i) The hyperplane H is ρ' -excellent, ρ' -good, (ρ', χ) -good (for $\chi(G_H) = 1$), (ρ', χ) -acceptable (for $\chi(G_H) \neq 1$) if and only if $\rho' = \beta_{\underline{r}, \rho}$ is of the form
 - a) $\underline{r} = (0, \dots, 0, r, 0, \dots, 0)$ and $\rho = 1$ or $\rho = \varepsilon$ or ρ is the standard representation or $\rho = (2, 2)$ if $r = 4$.
 - b) $\underline{r} = (0, \dots, 0, 1, 0, \dots, 0, r-1, 0, \dots, 0)$ and $\rho = 1$.
 - c) $\underline{r} = (0, \dots, 0, r-1, 0, \dots, 0, 1, 0, \dots, 0)$ and $\rho = 1$.
- (ii) H is (ρ', χ) -good (for $\chi(G_H) \neq 1$) if and only if $\underline{r} = (0, \dots, 0, r, 0, \dots, 0)$ and $\rho = 1$.
- (iii) The hyperplane H is (ρ', χ) -acceptable (for $\chi(G_H) = 1$) if and only if ρ' is one of the following representation
 - a) $\underline{r} = (0, \dots, 0, r, 0, \dots, 0)$ and $\rho = 1$ or $\rho = \varepsilon$ or ρ is the standard representation or $r \leq 5$ or $\rho \in \{(3, 3), (2, 2, 2), (4, 2)\}$ if $r = 6$.
 - b) $\underline{r} = (0, \dots, 0, 1, 0, \dots, 0, r-1, 0, \dots, 0)$ and $\rho = 1$ or $\rho = \varepsilon$ if $r \in \{3, 4\}$ or ρ is the standard representation if $r = 3$.
 - c) $\underline{r} = (0, \dots, 0, r-1, 0, \dots, 0, 1, 0, \dots, 0)$ and $\rho = 1$ or $\rho = \varepsilon$ if $r \in \{3, 4\}$ or ρ is the standard representation if $r = 3$.
 - d) $\underline{r} = (0, \dots, 0, 2, 0, \dots, 0, 3, 0, \dots, 0)$ and $\rho = 1$.
 - e) $\underline{r} = (0, \dots, 0, 3, 0, \dots, 0, 2, 0, \dots, 0)$ and $\rho = 1$.
 - f) $\underline{r} = (0, \dots, 0, 2, 0, \dots, 0, 2, 0, \dots, 0)$ and $\rho = 1$ or $\rho = \varepsilon \otimes 1$ or $\rho = 1 \otimes \varepsilon$.
 - g) $\underline{r} = (0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$ and $\rho = 1$.

Corollary 46 – Diagonal hyperplanes. Let $d \geq 2$, $r \geq 3$, H be a diagonal hyperplane of $G(d, 1, r)$ and $\rho' = \beta_{\underline{r}, \rho}$ be an irreducible representation of $G(d, 1, r)$ with $\underline{r} = (n_0, \dots, n_{d-1})$ and χ a linear character of $G(d, 1, r)$.

- (i) The hyperplane H is ρ' -good if and only if $\dim \rho \left(\sum_{j=0}^{d-1} \frac{j n_j}{r} \frac{r!}{n_0! \dots n_{d-1}!} \right) < d$
- (ii) The hyperplane H is ρ' -excellent if and only if ρ' is one of the following representation
 - a) $\underline{r} = (r, 0, \dots, 0)$;
 - b) $\underline{r} = (0, 0, \dots, 0, r, 0, \dots, 0)$ and $\rho = 1$ or $\rho = \varepsilon$;
 - c) $\underline{r} = (r-1, 0, \dots, 0, 1, 0, \dots, 0)$ and $\rho = 1$ or $\rho = \varepsilon$.
- (iii) The hyperplane H is (ρ', χ) -good if $\dim \rho \left(\sum_{j=0}^{d-1} \frac{j n_j}{r} \frac{r!}{n_0! \dots n_{d-1}!} \right) < d - n_H(\chi)$.

4.4 The group G_4

The group G_4 named after the Shephard and Todd classification [13] is a rank 2 reflection group. There is only one class of hyperplanes which is of order 3. The linear character of G_4 are given by the trivial one, the determinant and the square of the determinant. As an abstract group, G_4 is nothing else but $SL(2, \mathbb{F}_3)$. The irreducible representations of G_4 are then given by the 3 one-dimensional representation, the standard reflection representation named V , and

two others 2-dimensional representation $V \det$ (whose character is real) and $V \det^2$ and one 3-dimensional representation (whose character is real).

Corollary 47 – Hyperplanes of G_4 . Let H be an hyperplane of G_4 , ρ be an irreducible representation of G_4 .

Then H is ρ -excellent, ρ -good, (ρ, \det) -acceptable and $(\rho, 1)$ -good when $\dim \rho = 1$ or $\rho = V$ or $\rho = V \det^2$.

The hyperplane H is (ρ, \det) -good if and only if $\rho = 1$,

The hyperplane H is (ρ, \det^2) -good if and only if $\rho = 1$, $\rho = \det^2$ or $\rho = V \det^2$.

The hyperplane H is $(\rho, 1)$ -acceptable and (ρ, \det^2) -acceptable for every ρ .

4.5 The group G_5

The group G_5 named after the Shephard and Todd classification [13] is a rank 2 reflection group. There are two classes of hyperplanes which are both of order 3. In fact,

$$G_5 = \{j^k G_4, j = \exp(2i\pi/3), k \in \{0, 1, 2\}\} = G_4 \times \{\text{id}, j\text{id}, j^2\text{id}\}.$$

The irreducible representations of G_5 are then given by tensor product of representation of G_4 and of the three one-dimensional representation of $\{\text{id}, j\text{id}, j^2\text{id}\}$ which are given by $(1, \det, \det^2)$. One class of hyperplanes is in fact the class of hyperplanes of G_4 . The other one is a new class.

Corollary 48 – The hyperplanes of G_5 which are in G_4 . Let H be an hyperplane of G_5 which is an hyperplane of G_4 , ρ be an irreducible representation of G_5 and χ a linear character of G_5 . Write $\rho = \rho_1 \otimes \rho_2$ (resp. $\chi = \chi_1 \otimes \chi_2$) where ρ_1 (resp. χ_1) is an (resp. one-dimensional) irreducible representation of G_4 and ρ_2 (resp. χ_2) an irreducible representation of $\{\text{id}, j\text{id}, j^2\text{id}\}$

Then H is ρ -excellent if and only if H is ρ_1 -excellent for G_4 .

The hyperplane H is ρ -good if and only if H is ρ_1 -good for G_4 .

The hyperplane H is (ρ, χ) -acceptable if and only if H is (ρ_1, χ_1) -acceptable for G_4 .

The hyperplane H is (ρ, χ) -good if and only if H is (ρ_1, χ_1) -good for G_4 .

Corollary 49 – The hyperplanes of G_5 which are not in G_4 . Let H be an hyperplane of G_5 which is not an hyperplane of G_4 , ρ be an irreducible representation of G_5 and χ a linear character of G_5 .

The hyperplane H is ρ -excellent if and only if H is ρ -good if and only if $\dim \rho = 1$, $\rho = V \otimes 1$, $\rho = V \otimes \det^2$, $\rho = V \det \otimes \det$, $\rho = V \det \otimes \det^2$, $\rho = V \det^2 \otimes 1$, $\rho = V \det^2 \otimes \det$.

If $\chi = \det^i \otimes \det^k$ with $i+k = 0[3]$, then H is (ρ, χ) -good if and only if $\dim \rho = 1$, $\rho = V \otimes 1$, $\rho = V \otimes \det^2$, $\rho = V \det \otimes \det$, $\rho = V \det \otimes \det^2$, $\rho = V \det^2 \otimes 1$, $\rho = V \det^2 \otimes \det$.

If $\chi = \det^i \otimes \det^k$ with $i+k = 1[3]$, then H is (ρ, χ) -good if and only if $\rho = \det^{i'} \otimes \det^{k'}$ with $i' + k' = 0[3]$ or $i' + k' = 1[3]$ or $\rho = V \otimes 1$ or $\rho = V \det \otimes \det^2$ or $\rho = V \det^2 \otimes \det$.

If $\chi = \det^i \otimes \det^k$ with $i+k = 2[3]$, then H is (ρ, χ) -good if and only if $\rho = \det^{i'} \otimes \det^{k'}$ with $i' + k' = 0[3]$.

If $\chi = \det^i \otimes \det^k$ with $i+k = 0[3]$ or $i+k = 1[3]$, the hyperplane H is (ρ, χ) -acceptable for every ρ .

If $\chi = \det^i \otimes \det^k$ with $i+k = 2[3]$, the hyperplane H is (ρ, χ) -acceptable if and only if $\dim \rho = 1$, $\rho = V \otimes 1$, $\rho = V \otimes \det^2$, $\rho = V \det \otimes \det$, $\rho = V \det \otimes \det^2$, $\rho = V \det^2 \otimes 1$, $\rho = V \det^2 \otimes \det$.

4.6 The group G_{24}

The group G_{24} named after the Shephard and Todd classification [13] is a rank 3 reflection group. There is only one class of hyperplanes which is of order 2. The linear character of G_{24} are given by the determinant and the trivial one. As an abstract group, G_{24} is nothing else but the product of the simple groups $GL(3, \mathbb{F}_2) \times \{-1, 1\}$. Let us denote by 1 and ε the irreducible representations of $\{-1, 1\}$ and 1, $3_1, 3_2, 6, 7, 8$ the irreducible representations of $GL(3, \mathbb{F}_2)$ (determined by their dimension). The irreducible representations of G_{24} are then given by the tensor products of an irreducible representation of $GL(3, \mathbb{F}_2)$ and $\{-1, 1\}$.

Corollary 50 – Hyperplanes of G_{24} . Let H be an hyperplane of G_{24} .

The hyperplane H is ρ -excellent, ρ -good, $(\rho, 1)$ -good and (ρ, \det) -acceptable if and only if $\rho = 1 \otimes 1$, $\rho = 1 \otimes \varepsilon$, $\rho = 3_1 \otimes \varepsilon$ and $\rho = 3_2 \otimes \varepsilon$.

The hyperplane H is (ρ, \det) -good if and only if $\rho = 1 \otimes 1$.

The hyperplane H is $(\rho, 1)$ -acceptable if and only if $\rho = 1 \otimes 1$, $\rho = 3_1 \otimes 1$, $\rho = 3_2 \otimes 1$ and $\rho = 6 \otimes 1$ and $\rho = 1 \otimes \varepsilon$, $\rho = 3_1 \otimes \varepsilon$ and $\rho = 3_2 \otimes \varepsilon$ and $\rho = 7 \otimes \varepsilon$.

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